Lab 1: Asymptotes

Have you ever wondered about the future? What if you continue to grow for the next few years, how tall would you be? Suppose the latest flu continues to spread, how many people will become infected? If the cost of college continues to change, what will it be by the time you graduate? What does it mean for future students?

To answer these questions, in this lab we will investigate two mathematical concepts:

- What is the long term behavior of the phenomenon?
- If one can make a change in some aspect of the system, how will this affect the outcome?

To begin, we need the concept of an asymptote. A line is an asymptote for a curve if the distance between the line and the curve approaches zero as we move out farther and farther along the line. If the function has a horizontal asymptote, the graph "levels off" in the long run, close to a horizontal line, \( y = L \). In other words, horizontal asymptotes give us information about long term behavior.

For a vertical asymptote, the graph of \( f(x) \) approach get very large (or very small) as the values of \( x \) get close to \( a \). A vertical asymptote has an equation of the form \( x = a \). Finally, a linear slant asymptote has a nonzero constant slope and can be written as \( y = mx + b \) (with \( m \neq 0 \)).

Below are the graphs of a few function that appear to approach a fixed line as the value of \( x \) gets larger. Draw any line you think is an asymptote in each of the graphs.

![Graphs](image)

(a) ![Graph](image) (b) ![Graph](image) (c) ![Graph](image)

Figure 1

Check your lines in Figure 1, and label each as a horizontal, vertical or slant asymptote.
The behavior of the graph of \( f(x) \) depends on the function, which in turn depends on the values of the constants in the representation of the function. These constants are called parameters.

We will see what happens as we change the parameters, and what the long-range effect of these changes will be. A parameter represents a constant that when changed forms a new function. In the general form of the equation of a nonvertical line, \( y = mx + b \), \( m \) and \( b \) are parameters.

They represent the slope and \( y \)-intercept of a line, respectively. For any particular line, these parameters have constant, numerical values. The function is a function of \( x \), whose domain is all real numbers.

Select the Function Kit and open the Asymptote Tool.

**Tool Instructions:**

As you move the cursor over the graph, the values of \( x \) and of \( f(x) \) change. If you click on the trace button, you will be able to read the coordinates of the points as you move the cursor over the graph.

If you wish to see the asymptotes displayed (if any), click on the asymptote button.

To change the parameters in a function family, use the cursor to move the slider. The function with the new parameter will be displayed. Note that parameters are the values you can change with a slider.

There is a special trick for looking at the history of the functions with different parameters. When you click above the slider (in the strip where the numerals are located), the functions will be displayed for various values of the parameter, and the values of the parameters will be indicated in the top, right window. The values are color-coded to match the graphs represented. After six values appear, the window will be reset.
Click in upper left corner to quit and return to the main menu screen.

To select a new function from a menu of functions, click the mouse in the function box arrow (on the right of the screen.)

Now, we will examine some functions which have asymptotes and discuss how an asymptote is affected by the parameter values present in the function.

The opening screen shows the graph of \( f(x) = \frac{a(1+x+x^2)}{x^2} \). Move the slider and observe what happens to the graph. Click above the line and notice that the values of \( a \) are shown in the upper box, in the same color as the graph. How does the value of \( a \) affect the graph?

Watch the graph change as you change \( a \). Guess where the horizontal asymptote will occur?

To see the asymptote with the graph, click on the **asymptote** button. Does this information verify your guess?

We begin by trying one problem step-by-step before letting you experiment on your own. Choose the function \( f(x) = c/(a + (1-a)e^{-bt}) \), function #8 which is called a **logistic** function. Fix \( a = 0.1 \). Fix \( c = 1.5 \). Examine the graphs for \( b = 2, 3, 4, 5, 6 \). As \( b \) increases, describe how the graph of \( y = f(t) \) changes. What can you say about the effect of the coefficient \( b \)?

Again, using function #8, this time fix \( b = 4, c = 1.5 \). Examine the graphs for \( a = 0.2, 0.3, 0.4, 0.5, 0.6 \). As \( a \) increases, describe how the graph of \( y = f(t) \) changes. What can you say about the effect of the coefficient \( a \) on the graph?
1. Simple monomials

Before considering other applications, let us consider a simple one-parameter family of functions, \( f(x) = ax^n \), where the parameters are \( a \) and \( n \). Each value of \( n \) gives a different function with a different graph, but all of the graphs will have some common features. When \( n \) has a fractional value, not all values of \( x \) are in the domain of the function, so be careful! We will discover how the value of \( n \) affects the graph of \( f(x) = ax^n \). Begin by setting \( a = 1 \).

1.1 Let \( a = 1 \), and using integer values for \( n \), describe the values of \( n \) for which the graph have a horizontal asymptote?

1.2 What is the equation of the asymptote?

1.3 For what values of \( n \) does the graph have a vertical asymptote?

1.4 Repeat these exercises for \( a = 2 \).
1.5 Describe completely the graph for values of \( n < 0 \).

1.6 Try several different values of \( a \) and describe any effect on the graph and the asymptote.

1.7 Now let \( a = 1 \) again and try fractional values of \( n \): \( n = 1/4, 1/2, 3/4 \) and \( n = -1/4, -1/2, -3/4 \). Describe the pattern you see. (Does the graph have any asymptotes? Is the function defined for all values of \( x \)?)

2. Rational functions
Let us now consider the graph of a few rational functions, \( f(x) = \frac{g(x)}{h(x)} \) and the effect of parameters on these graphs. Notice that we must exclude from the domain values of \( x \) for which the denominator is zero. In many cases, a vertical asymptote can occur at an excluded value.

The problem below illustrates the importance of rational functions. Recall that a linear cost function can be expressed in terms of the fixed cost, \( a \), plus the cost per item, \( b \), times the quantity, or number of items, produced \( x \). The average cost, \( A(x) \), is the total cost divided by the quantity produced.

In symbols, a linear cost function can be written as \( C(x) = a + bx \). Write the average cost function in terms of the cost function:

\[
A(x) = ______________________
\]
2.1 Production costs: Select the average cost function from the list. Set the fixed cost slider to be 100.

2.1.1 As the number of items produced increases, describe what happens to the average cost per item.

2.1.2 Is there a limit to how small the cost per item can become? If so, give the general equation for the horizontal asymptote.

2.1.3 Now vary the values of \( a \) and determine the effect on the asymptote.

3. Exponential functions

Exponential functions (in which the variable appears in the exponent) are another class of functions which share some characteristics. In each of the problems below, again we are interested in the asymptotic behavior of the functions. Before tackling some applications, let us explore two exponential functions.
3.1 Choose the function \( f(x) = a^x \). Use the slider to examine the graph for several values of the parameter \( a \). Record your observations below for \( a = 1, \ a = 2, \ a = 0.5 \).

3.2 Determine under what conditions the graph has any asymptotes.

3.3 Now consider the function \( f(x) = \left(1 + \frac{1}{x}\right)^{ax} \). Again use the slider to examine the graph for several values of \( a \). Record your observations below. Determine if the graph has any asymptotes.

4. Familiarity with a product

When an announcement is made on television, many people become aware of it at one time. Sometimes, a product is introduced nationally on the radio or on television. Let \( y \) represent the number of people who have heard of the product at time \( t \) through these media. A function which represents the spread of information about the product is 
\[
y(t) = M(1 - e^{-at})
\]

Click on this function to view the graph.

The parameter, \( a \), represents the percentage of the population who heard the announcement on day 1. The total population, in millions, is represented by \( M \).

4.1 Describe the function for \( a = 10\%, \ 20\%, \ 	ext{etc.} \) of the population and a total population \( M = 2 \) (million). [Set the \( a \) slider to 0.10, 0.20, etc.]
4.2 Use the sliders to determine the relationship among $a$, $M$, and the asymptote for the graph. Describe what you find.

5. **Vertical and slant asymptotes**

   The examples above emphasized horizontal asymptotes. However, in Figure 1, there were three types of asymptotes presented: vertical, horizontal, and slant asymptotes. Let us concentrate the effect of a parameter on a vertical asymptote.

   5.1 In the list of functions, find a family that has a vertical asymptote. Give the function and its vertical asymptote in space below.

   5.2 Look over the list of functions. By studying the graphs, find a family of functions that have slant asymptotes. On the axes below, sketch the graph of the function and its asymptote for two different values of the parameter.

   5.3 Does the parameter affect the slant asymptote? If so, explain how.
6. Further exploration

Curves can exhibit asymptotic behavior where the asymptotes are not straight lines. An example of this and other further explorations are given below.

Note that there are several functions in this list of the same type. See if you can determine which ones are from the same family.

**General cost functions:** As you progress through the applications given in this course, you will notice that many (if not most) cost functions are cubic functions, of the type

\[ C(x) = ax^3 + bx^2 + cx + d \]

For this type of function, the average cost of producing \( x \) items is given by

\[ A(x) = \frac{ax^3 + bx^2 + cx + d}{x} \]

where \( d \) is the fixed cost. Note that having this fixed cost gives us a vertical asymptote when \( x = 0 \). In this example, we only consider one particular cubic cost function, namely the function which describes the cost of college tuition: \( f(x) = 0.2x^3 + ax^2 + 2a^2x + 6 \). Find the average cost function.

Graph this function for various values of the parameters and determine if the resulting graphs have any familiar characteristics. That is, can you describe a function which is an asymptote for the graph?
7. Additional Problems

7.1 Learning a task

When a task (such as stuffing envelopes) is practiced $x$ times, the time it takes to complete a single task is given by $t = \frac{a(1+x+x^2)}{x^2}$ where $t$ is in seconds, and $x$ represents the number of times one has practiced. Click the $f(x)$ button and choose the function represented above. Use this function to answer the following questions. Let $a = 10$ to answer the following question.

If you practice only once, e.g. stuff one letter into an envelop, how long does it take you? [Notice that $x$ can only be an integer in this example. In general, however, $x$ can have fractional values. The function given agrees with the function having only integer values of $x$ whenever the $x$ value is an integer.]

If you are on a committee, stuffing flyers into envelops for distribution on campus, how long would it take you to stuff each of 25 envelopes?

If there a limit to how fast you can stuff an envelop? If yes, give plausible explanations for this.
Is there a horizontal asymptote for this graph? If yes, what is the horizontal asymptote?

Vary $a$ and determine what effect the value of $a$ has on the asymptote. Explain what this means.

7.2 More General Production costs

A simple general model for the average cost (in dollars) of producing $x$ batches of 100 items is modeled by $A(x) = \frac{a+bx}{x}$. Choose the function $A(x) = \frac{a+bx}{x}$. Try different values of the parameters, $a$ and $b$. Use your calculator (or calculate by hand) the cost of producing one batch in terms of the parameters?

What happens to the average cost of producing $x$ batches as the number of batches get larger? Does the graph have any asymptotes? If so, interpret these values?

What effect, if any, do $a$ and $b$ have on the asymptote?
For example, the average cost of producing $x$ batches of one hundred gallons of gasoline has the parameters $a = 1130$ and $b = 0.06$. Determine the cost of various numbers of batches. Estimate a value of $x$ for which the average cost is within $0.01$ of the limit.

7.3 Growth of Sales of a New Product

The function $y = M(1 - e^{-at})$ also can be used to model the growth of sales of a new product. If $a$ represents the total pool of persons available to purchase the item, and $a$ represents the proportion of these people reached each day by salespeople, describe what information is given by the function.

If you want to make a "blitz" on the market to get many sales in the shortest possible time, what strategy should you use?
7.4 Memorization, an Example from Learning Theory

The number $y(t)$ of words that can be memorized in $t$ minutes can be determined using a model which has limited growth, such as the model $y = M(1 - e^{-at})$, where $M$ is the total number of words to be memorized. Suppose a person can memorize a list of words at the rate of 15 words in 25 minutes. Can we determine how long it will take to memorize 100 words? 200 words? In this example, $M = 100$ or $200$. Click on the function in the list, and answer the following questions.

First determine the rate per minute, or the number of words a person can learn in one minute. In this example, the rate is $\frac{15}{25} = 0.60$ words per minute. The rate of learning is given by $a$ in the formula. Substitute the value of $a$ and for each example, $M = 100$, $M = 200$, determine how long it will take to memorize the words.

Determine if there is a horizontal asymptote on the graph. If so, describe the asymptote.

What affect do the parameters, $a$ and $M$, have on the asymptote?

7.5 Spread of the Flu

An infectious disease, like the flu, is transmitted from one infected person to a noninfected person. The model representing the proportion of people infected at a particular time is $f(t) = \frac{c}{a + (1 - a)e^{-b}t}$, where $a$ is the initial proportion infected, $b$ is the transmission rate, or the rate of contact with the initial number infected. Using the graph of $f(t)$, try various values of $a$ and $b$ to determine the effect of changing the initial proportion infected and the rate of transmission respectively. Record what you observe.
Determine any horizontal asymptote, and describe the effects of \( a \) and \( b \) on the graph of the function in terms of the asymptote, intervals of population infection. Determine the maximum proportion of the population infected.

7.6 Lyme disease

If an individual is infected with a bacteria, the number of bacteria will increase to a point and (hopefully) eventually decrease in number. The function \( f(t) = te^{-bt} \) represents the number of bacteria (in millions) in a person \( t \) months from the initial infection. Determine the effect of changing the parameter, \( b \), on the graph of the function \( f(t) \).

The behavior of Lyme disease is described by a function of this type with \( b = .09 \), \( t \) measured in months. Describe the behavior of Lyme disease. After what length of time will there be less than 3 (million) bacteria in the body? (Hint: Roll the cursor over the graph to answer this question.) What is the long term prognosis for the disease?