Instructor Notes 15: Definite Integral

For a function $f(x) \geq 0$ on an interval, the area under a curve is expressed by the definite integral. In this lab, we explore the relationship between the area under the curve and the antiderivative of a function. In an applied calculus course, the emphasis is on understanding the concept of area under a curve, and what it represents. This lab is designed as a discovery lab experience wherein students see the relationship between the area under a curve and the antiderivative evaluated at certain values.

In the Definite Integral Tool, in the Integration Kit, the area under the curve is shown.

- In the opening screen, the $a$ slider is set on 0.
- As you move the $b$ slider to the right on the lower graph, you will see the area under the curve $y = f(x)$ appear under the curve (and above the x-axis).
- Click on the value of $b$ at a particular point. The value of the area for $x$ in the interval $[0, b]$ will be plotted in the upper graph. After you have plotted several points, you can see the graph of the area function, $y = F(x)$, by clicking on the Show F(x) button.

After introductory examples, students are given several applications. They will see examples wherein the function is negative over part of its domain. They are let to discuss what happens to the area (total and net areas) in this case.

Two examples that we did not include in this lab, but you may wish to introduce at this time are the present value of an income stream and probability functions. If your students are interested, below are two more sections you can use of exercises for your students.

Income Stream

A continuous income stream is the process by which a company receives income on an ongoing basis. The rate at which the income is generated varies from time to time. If $S(t)$ is the rate at which the income is generated, in dollars per year, we can find the Present and Future Values of the Income Stream. Assume that the money earned is deposited into an account which pays interest with rate $r$ compounded continuously. The present value of the income stream from the present to time $T$ years in the future is given by

$$\int_0^T S(t) e^{-rt} \, dt$$

Although not part of the lab, one can extend this concept to discuss the future value of an income stream at time $T$, which is given by

$$\int_0^T S(t) e^{(T-t)} \, dt$$
Suppose you are the owner of a small company, and plan to take over another company in five years. To do this, you will need $50,000 in cash, and you plan to finance the rest.

We want to find income stream generated from $1200 per year for the next 10 years if the interest rate is 6%. Write the expression to be integrated using the information above. In this case, $S(t) = 1200$ per year.

\[
\int_0^{10} S(t)e^{r(T-t)}\, dt
\]

Find the present value of the income stream for the first ten years by using the tool.

Find the present value of the same income over the next 20 year.

If a company receives a stream of income over a period of time, and reinvests the income at a fixed rate, \( r \), per year, we can determine the value of this money at some future time. The future value of an income stream at time \( T \) is given by

\[
\int_0^{T} S(t)e^{r(T-t)}\, dt
\]

Suppose you are the owner of a small company. After expenses, you expect to generate $20,000 in revenue per year that can be reinvested in a business earning 10% per year compounded continuously. Write the integral to represent this information.

Find the value of this investment after five years.

Suppose you are making a long range plan, how much will the investment be worth after 10 years?
How much did the investment grow in the period from $T = 5$ to $T = 10$?

**Probability functions**

In statistics, a variable which represents a phenomenon is called a *random variable*. If the variable, $x$, can assume all values in an interval, it is called a *continuous random variable*. For example, the age of an individual from the time of birth, $x = 0$ to the present is a continuous random variable. The length of time spent waiting in a queue (checkout counter line, etc.) is also a continuous random variable. The probability, or likelihood, that a random variable takes on a range of values, $a \leq x \leq b$, is given by the area under curve of a function which represents the graph of the data. In general, if $f(x)$ is a probability density function, then the probability that $a \leq x \leq b$, expressed as $P(a \leq x \leq b)$ is given by

$$P(a \leq x \leq b) = \int_{a}^{b} f(x)dx$$

These functions have a total area under the curve equal to 1, or 100%.

The life span (in hours) of an ordinary 60 watt light bulb is given by a continuous random variable, $x$, with probability density function $f(x) = .0001e^{-0.001x}$. Select this function from the list. Find the probability that a light bulb lasts less than 500 hours. (Note: That is today, find $P(0 \leq x \leq 500)$).

Find the probability that the light bulb lasts between 500 and 1000 hours.