Lab 15: Definite Integral

Jan. 25, 2001

What is the connection between accumulation (discussed in another lab) and area between a curve and the horizontal axis? Is there a connection between "antiderivative" and accumulation? Things are beginning to add up, as you will see in this lab.

If a function has only positive values in an interval, \([a, b]\), then the area between a curve \(y = f(x)\) and the \(x\) - axis is expressed by the definite integral, \(\int_{a}^{b} f(x)\,dx\). Although this seems like a strange notation, the meaning of the notation is explained in the lab "Numerical Integration." In this laboratory, we explore the relationship between the area under the curve, accumulation, and the antiderivative of a function.

Open the **Definite Integral Tool** in the Integration Kit

- In the opening screen, the \(a\) slider is set on 0.

- As you move the \(b\) slider to the right on the lower graph, you will see the area between the curve \(y = f(x)\) and the \(x\)-axis from the point where \(x = 0\) to \(x = b\).

- Click on the value of \(b\) at a particular point. The value of the area for \(x\) in the interval \([0, b]\) will be plotted in the upper graph. After you have plotted several points, you can see the graph of the area function, \(y = F(x)\), by clicking on the Show F(x) button.

1. **Finding the area of a region**

1.1 Select the function \(f(x) = 0.75x^2\). Find the area bounded by the \(y\)-axis, \((x = 0)\), the \(x\)-axis, and the line \(x = b\) for the function, \(y = 0.75x^2\) for the given values of \(b\):

<table>
<thead>
<tr>
<th>(b)</th>
<th>Area under the curve from (x = 0) to (x = b)</th>
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<tbody>
<tr>
<td>0.5</td>
<td>_______________________________</td>
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<tr>
<td>1.0</td>
<td>_______________________________</td>
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<tr>
<td>1.5</td>
<td>_______________________________</td>
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<tr>
<td>2.0</td>
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</table>

1.2 Describe the graph of the area function in the upper window.
1.3 What do the green points in the upper window represent?

1.4 When you roll the cursor over the graph of $y = F(x)$, you see lines tangent to the curve. What function represents the slopes of these tangent lines?

1.5 Can you discern a relationship between the function which represents the area, $F(x)$, and the function $f(x)$?

1.6 Now, fix the value of $b$, $b = 2$. Move the $a$ slider and find the area bounded by $x = a$, $x = b$, $y = 0$, and the curve, $y = f(x)$. Record the values of the area in the chart below.

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<tbody>
<tr>
<td>$a$</td>
<td>0.2</td>
<td>0.4</td>
<td>0.6</td>
<td>0.8</td>
<td>1.0</td>
<td>1.25</td>
<td>1.5</td>
<td>1.75</td>
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<td>Net Area</td>
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1.7 Repeat exercise 1.6 for $b = 4$ and $a = 0, 1, 2, 3$.

1.8 Did you notice that the function representing the area from $x = 0$ to $x = b$ under the curve is given by a function $F(x)$ whose derivative is $f(x)$? To find the area from $x = 0$ to $x = b$, we can find $F(x)$ from $f(x)$ and then (assuming $F(0) = 0$) substitute $x = b$ to find $F(b)$. With this information, how would you expect to find the area under the curve from $x = a$ to $x = b$? State your hypothesis.

1.9 Assume $F(0) = 0$, and verify your hypothesis using two different functions in the list. Find the area under the curve for several intervals $[a, b]$.

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$F(x)$</th>
<th>$a$</th>
<th>$b$</th>
<th>$F(a)$</th>
<th>$F(b)$</th>
<th>$F(b) - F(a)$</th>
<th>Net Area</th>
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1.10 Consider the following functions that are not on the list. Estimate the net area between the curve and the $x$-axis.

![Graph of two functions with axes labeled and ranges from -1 to 6 on the x-axis and -2 to 3 on the y-axis.]

**upper graph**

**lower graph**

2. **Area as total income**

At some colleges and universities, tuition is paid in installments over a year, rather than as one lump sum at the beginning of the semester. The total revenue to the college is the amount of tuition per student multiplied by the number of students. For simplicity, let us consider the total tuition expected from 100 students over the next five years. Under this installment condition, we can consider the income that college receives as a continuous income. At time $t$, $f(t)$ represents the amount of tuition obtained at that time from the 100 students. Total tuition income up from the start of the fiscal year, $t = 0$ to time $t = b$ is the area under the curve in this interval.

Generally, this income is expressed as nonnegative cubic function. Before calculating the total tuition income to the institution, let us look at more general cubic function.

2.1 Choose the function $f(t) = t^3 - 6t + 9t - 1$ from the function list.

2.1.1 What is the value of the area under the curve for $0 \leq x \leq 0.15$? Note that the area is below the $x$-axis. The $f(x)$ value are negative, and the area is negative. Hence, the function $F(x)$ represents the net area.

2.1.2 To see this, move the $a$ slider to $a = 0.15$. Notice that the value of the area increases. This is because the negative area has been subtracted. With $a$ fixed, try
several values of $b$ until you are able to show the graph. Why is the function decreasing on the right?

2.2 Now, let us consider a function that could represent college tuition income. For example, the function $f(t) = 100(0.4t^3 - 2t^2 + 4t + 12)$. Use the fact that the function representing total tuition income is $F(x)$, where $F'(x) = f(x)$.

2.2.1 Starting now ($t = 0$), find the total tuition income for the next two years.

2.2.2 Find the total income from tuition from these students in the fourth year (i.e., from $t = 3$ to $t = 4$).

2.2.3 Find the total tuition from these 100 students over the a four year period.

2.2.4 If 100 students transfer to this college two years from now, and spend the following two years at the college, what will be the total revenue from these students? (NOTE: $2 \leq t \leq 4$.)
3. Area as sales revenue over time

A new product "Hot Stuff" is planned to be released into the market just in time for the Christmas season. From past experience, the sales at time $x$ weeks after introduction of a "hot" item behave according to the function $f(x) = te^{-x}$.

3.1 What will the total revenue be from the sales of a Hottie over the first week, $0 \leq x \leq 1$, from its introduction?

3.2 If the item is introduced at Thanksgiving time, four weeks before Christmas, what will be the revenue between the time it is introduced and Christmas? $[0 \leq x \leq 4]$?

3.3 After the holiday season, sales of Hottie will begin to slow. Using the model for sales, what will be the total revenue in the month after Christmas? $(4 \leq x \leq 8)$?
4. Further Explorations

In most of the examples we considered, the area considered was above the $x$-axis. This may not always be the case. If part of the area lies below the horizontal axis, the definite integral will not represent the total area, but rather the net area. The reason for this is explained in the Numerical Integration lab. To find the total area, one must divide the interval into subintervals, and find the area in the subintervals individually, then find the total of the areas in the subintervals. In the figure below, what subintervals would you choose?

A few functions in the list have characteristics which make them different from the others we have explored. This idea was mentioned earlier. See if you can see what is different in each case. Also see if you can find the area under the curve in each instance using the definite integral. Explain why or why not for each case.