Instructor Notes 5: Tangent and Slopes

Purpose

The **Tangent Slopes Tool** aids students in visualizing the relationship between the value of the derivative, the slope of the curve, and the graph of the function. The graph of \( y = f'(x) \) appears directly below the graph of \( y = f(x) \) so that students can see corresponding points, make inferences about the relationship between \( f(x) \) and \( f'(x) \) and immediately verify these inferences or correct their misconceptions.

Prerequisites

- Knowledge of the derivative as representing the slope of a curve.
- The concept of increasing and decreasing function.
- Ability to calculate a derivative.

Concepts developed

With this tool we explore the relationship between the graph of a function, its "steepness" at various points, its functional expression, and its derivative. Students will see patterns between the graph of the function, and the graph of its derivative; visualize steepness of a curve and its relationship to the value of the derivative at those points.

You may use this tool to have students trace the tangent to the curve and to discuss the characteristics of a graph in relationship to the derivative. For example, the slope of the curve at various points can be used to determine the steepness of the curve, when the function is increasing, decreasing, or achieving an extrema. Points where the graph of \( y = f'(x) \) crosses the \( x \)-axis are aligned with the critical points on the graph of \( y = f(x) \). Looking at the lower graph, one can determine when the derivative is positive, when it is negative, and when it is zero thus determining when the function is increasing, decreasing, or neither. Students see immediately the correspondence between these characteristics of the derivative and the characteristics of the function \( f(x) \).

With the Tangent Slopes tool, we estimate the value of the derivative at various points, plot points for \( f'(x) \), and compare the actual derived function to the estimated function. The graph of the derivative can be shown if six points have been plotted. A good example is to show that the derivative of \( e^x \) is \( e^x \) by plotting points in the lower graph (representing values of the derivative) for various values and \( x \) and then comparing the graph of \( y = e^x \) with the graph of \( y' = e^x \).

Tool Instructions

Open the Derivative Ket and choose the **Tangent Line Slopes** tool. The default graph in this tool is designed to look like a hypothetical path of a section of a roller coaster. To explore the characteristics of the function, open the Tangent Slopes tool. Below is the function and the graph which appear when the tool is opened.
As you roll the cursor over the upper graph, the tangent to the curve appears and the coordinates of the points are given. In the lower window, the graph of the derivative can be plotted by clicking on points in the plane.

**Buttons**

- Click on the button between the graphs to show or hide the lower graph, the graph of \( f'(x) \).
- Roll the cursor over the upper graph to see the tangent lines to the curve.
- To plot points which estimate the value of the slope of the tangent line, place the cursor on the lower graph, which is the graph of \( y = f(x) \). Drag the cursor with the mouse on the lower graph until the line on the upper graph of \( y = f(x) \) is tangent to the curve. Click to set the point, representing the \( x \) coordinate and the slope of the tangent line at that \( x \) value.
- To find a tangent at a particular \( x \) value, move the cursor in the lower window vertically until the line appears to be tangent to the curve in the upper window. To record the value in the table, click the mouse when the line appears to be tangent to the curve.
- Click in the \( x \) column of the table to type in values.
- Click "sort" on any column to sort from smallest to largest values. The entire table is sorted in ascending order relative to the values in the column chosen.
- Click in the lower graph to add more points to the graph.
- After estimating at least six values for \( f'(x) \), you may click **Graph** \( f'(x) \) to see the derivative of the function.
- Click "\( f(x) \)" to select a different function.

After plotting points, you can readjust those for which the line in the \( f(x) \) window was not a tangent line; the value in the table will be replaced automatically. (If you do not have a good match to the curve, click [clear] and try again.) You may add values of \( x \) to your table by clicking on the \( x \) column of the table and entering a value. The tool will calculate the \( f(x) \)
and $f'(x)$ values. [Values may be numerical, e.g. $-1$, $3$, or multiples of $\pi$ in the interval, e.g. $.5\pi$, $.25\pi$, etc.]

The last two functions in the list are there to demonstrate the pixel problem with to show that even though the graph of $y = f(x)$ looks pretty smooth, there are often great changes that we cannot see. We can, however, see these in the derivative function.
Sample classroom demonstration - Tangent and Slopes

Note: The following pattern of questions and comments can be used for those functions of interest for a particular topic in the course.

Choose a function type from the function type list, and then select a particular function from the next menu, or use the function which appears when the tool opens.

Plot at least six points on the lower graph, \( f'(x) \), representing a variety of \( x \) values across the domain. When you have completed six entries, you will be able to see the graph of \( y = f'(x) \) by clicking the mouse on the key: Graph \( f'(x) \).

When the table is complete, click **Sort** on the column of \( x \) values to arrange the table in increasing values of \( x \).

If you do not have a slope of zero in the \( f'(x) \) column of your table, you can roll the cursor over the graph of \( y = f(x) \) in the top window and view the value of the slope of the tangent lines. Record any \( x \) coordinate that you find and the corresponding \( f(x) \) value. You may also add a value of \( x \) to your table by clicking on the \( x \) column of the table and entering a value. The tool will calculate the \( f(x) \) and \( f'(x) \) values. [Values may be numerical, e.g. \(-1, 0, .3\), or multiples of \( \pi \) in the interval, e.g. \(.5\pi, .25\pi\), etc.]

Example 1: Locating points at which there is a specific value of the slope: Use the table of values to locate points for which the slope is 1. (There may be several.) If none appear in the table, discuss with the class how one can be found. Find the coordinates of these points. Repeat for other values of the slope, including negative values and zero.

[Note: You can add this tool to demonstrate the steepness of the curve by choosing any values of \( f'(x) \) you like. You may also use this tool to demonstrate the Mean Value Theorem by having the students find \( \frac{f(b)-f(a)}{b-a} \) and then finding the points with this slope by rolling the cursor over the upper graph.]

Example 2: Demonstrate points for which the slope is large, small, positive, negative and zero. Discuss the shape of the curve \( y = f(x) \) for each of these. Use the points with \( f'(x) = 0 \) to determine the local extrema of the function.

For some functions, the slope is undefined for \( x = a \). Discuss why this occurs and trace the graph for values close to this \( x \) value. Students will see the slopes increasing and observe that the \( \lim_{x \to a} f(x) = \infty \). Discuss similarities and differences between these functions with cusps and those with no cusps.

Example 3: From the table of values and the graph, determine the interval(s) of \( x \) values for which \( f(x) \) is increasing. Comment (or ask students) on the values of the \( f'(x) \) on these intervals. Repeat for the intervals on which the function is decreasing. Relate the characteristic of the curve to the values of the slopes.

Again using the graph, determine the values of \( x \) at which extrema occur. Find the points in the table of values (or add the values to the list). You can find the largest and smallest values by sorting on the \( f(x) \) column.

Reinforce this concept by finding the values of \( x \) for which the derivative is zero or undefined. If your table does not contain points for which \( f'(x) = 0 \), add points to the table by inserting \( x \) values. Ask students to verbalize what is occurring.

If you want to illustrate increasing and decreasing characteristics, use the lower graph to determine when \( f'(x) > 0 \) or \( f'(x) < 0 \). Again, ask students to verbalize their observations.

Generalizations: Have students make generalizations from their observations. Use other functions to test their generalizations.