Lab 7: Inverse Functions and Their Derivatives

Certain functions have inverses. The inverse of a function will "undo" the action of the original function. The graphs of a function and its inverse have a beautiful symmetry across the line \( y = x \). The symmetry extends to the tangents to the curves. They are all heading your way in vivid color!

1. What is an inverse function?

Suppose \( f: A \to B \) is a function. We know that \( f \) is a rule that maps each element of \( A \) to a unique element of \( B \). Now if there is an inverse function \( f^{-1} \), it will have to map each element in the range of \( f \) to a unique element in \( A \) so that

\[
f^{-1}(f(x)) = x \quad \text{for each } x \text{ in } A.
\]

(Also we can see that \( f \) must be an inverse function for \( f^{-1} \).) We say that \( f \) and \( f^{-1} \) are an inverse function pair.

1.1 Show that \( f(x) = \sqrt{x} + 1 \) and \( g(x) = (x - 1)^2 \) are inverse functions, that is, for each \( x \) in \([0, \infty)\), \( g(f(x)) = x \) and for each \( x \) in \([1, \infty)\), \( f(g(x)) = x \).

2. What functions have inverses?

Notice the problem if \( f \) maps two different \( x \)-values, say \( x_1 \) and \( x_2 \), to the same \( y \) in \( B \). Then how can we construct an inverse function \( f^{-1} \) for \( f \)? We wouldn't know which \( x \)-value to pick to map back to. We can't choose them both because then \( f^{-1} \) would not be a function!

We get around this problem by selecting only those functions that are one-to-one:

A function \( f : A \to B \) is one-to-one if whenever \( x_1 \neq x_2 \) in \( A \), then \( f(x_1) \neq f(x_2) \) in \( B \).

2.1 Recall that the horizontal line test is a quick way to check a graph of a function to see if the function is 1-1. Just mentally check to see that every possible horizontal line can intersect the graph of the function in exactly one place. Check the graphs
that follow to see whether or not they are one-to-one. Label them and justify your answer.

Note that some functions can be made one-to-one by restricting their domains. For instance \( f(x) = x^2 \) is not one-to-one on \((-\infty, \infty)\) but it is on \([0, \infty)\).

3. Symmetry of inverse functions across the line \( y = x \)

3.1 Open the Inverse Function tool in the Derivative Kit. Pick any one of the first three pairs of inverse functions. Roll the cursor over the graphing plane. Notice how corresponding points are reflected across the line \( y = x \). That is, if \((a, b)\) is on one curve then \((b, a)\) is on the inverse curve. Explain why this is so in terms of what inverse functions do!

3.2 For each function graphed in Figure 2, sketch its inverse function by reflecting the curve across the line \( y = x \).
3.3 Now click on the selection box for **Tangents**. Note that tangents to the two curves have the same property of reflecting across the line \( y = x \). State the functions you selected (specify constants \( a \) and \( b \) if appropriate). Sketch the curves and their tangents on the axes provided.

![Graph showing tangents reflective across the line y = x](image)

3.4. In the following true/false questions, use your knowledge about the domain and range of a function and your observation of the tool to answer the questions.

a. True or False: If \( f \) is continuous and increasing on an interval, then so is \( f^{-1} \) on the corresponding interval.

b. True or False: \( \text{Dom } f^{-1} = \text{Rng } f \)

c. True or False: \( \text{Rng } f^{-1} = \text{Dom } f \)

3.5 Select the third function from the function menu. Note that \( g(x) = (x - a)^2 \) is not one-to-one over its entire domain \( (-\infty, \infty) \). Yet it is one-to-one for the **Inverses**.
tool. What is the largest domain over which it is one-to-one (so that the domain includes the domain shown on the tool)?

3.6 What function on the Inverses tool is it own inverse for every value of \( a \)? Explain.

3.7 Select the function \( f(x) = \frac{a-x}{a+x} \). Using the tool, find the value of \( a \) for which the function is its own inverse and prove that this is the case by direct substitution.

4. The derivatives of inverse functions

We are going to illustrate the following theorem using the Inverses tool:

**Theorem for the Derivatives of Inverse Functions**

If a differentiable function \( f \) has an inverse function \( g = f^{-1} \) and \( f'(g(c)) \neq 0 \), then \( g \) is differentiable at \( c \) and

\[
g'(c) = \frac{1}{f'(g(c))}
\]

One way of interpreting this statement is to say that derivatives of the function and its inverse at corresponding points on their graphs are reciprocals of each other.

This process is easier to see on the Inverses tool than it is to describe. Click on the Secants selection box and clear the Tangents selection box to reduce the number of lines on the screen.

Recall that to find \( f'(x) \) we take the limit of the slope of the secant line from \((x+h, f(x+h))\) to \((x, f(x))\) as \( h \to 0 \). Notice that as we do this, the rise over the run for the corresponding points in the inverse curve are reciprocals relative to the original curve. Since the
ratios of rise over run are reciprocals at every corresponding point, the derivatives must also be reciprocals.

4.1 Let $f(x) = \sqrt{x} + 1$ so that $f^{-1}(x) = g(x) = (x - 1)^2$. So the point (4,3) is on the graph of $f$ and the corresponding point for $g$ is (3, 4). Use the theorem to calculate $g'(3)$.

5. Important inverse functions: exponential and logarithmic functions

An important inverse function pair is:

$$f(x) = \log_e x$$
$$g(x) = b^x$$

for $b > 0$ and $b \neq 1$.

Note that for $b = e$, we have the natural log and exponential functions:

$$f(x) = \log_e x = \ln x$$
$$g(x) = e^x$$

5.1 Move $b$ on the slider as close as possible to $e = 2.718...$, the base for the natural logarithms. Label the scale on the axes provided. Sketch the graph of the inverse function pair.