Equality of Mixed Partial (Clairaut’s Theorem)

In chapter 14, we encountered Clairaut’s Theorem, which says that when \( f_{xy} \) and \( f_{yx} \) are both continuous, they will be equal. This was presented as a fact—something to believe only because I told you so. Such teaching by dictatorship is always regrettable; a guiding principle of the discipline of mathematics is that it is democratic and egalitarian—that anyone can challenge anyone else’s statement, and logic will judge who is right. I had no choice, though; the justification of Clairaut’s Theorem would not have made sense to you at the time.

It is now possible to close this gap: using the perspective of §15.2, I can now back up the claim of Clairaut’s Theorem with a proof. I will break off the main part as a lemma.

**Lemma.** Let \( f_{xy} \) and \( f_{yx} \) be continuous on rectangle \( R = [a,b] \times [c,d] \). Then

\[
\int \int_R f_{xy} \, dA = \int \int_R f_{yx} \, dA = f(b,d) - f(a,d) - f(b,c) + f(a,c).
\]

**Proof.** I will prove the assertion for \( f_{xy} \); the proof for \( f_{yx} \) will be left as an exercise.

\[
\int \int_R f_{xy} \, dA = \int_a^b \left( \int_c^d f_{xy}(x,y) \, dy \right) \, dx
\]

(by the FTC) \( \rightarrow \) \[
\int_a^b \left( f_x(x,y) \bigg|_{y=d}^{y=c} \right) \, dx
\]
\[
= \int_a^b f_x(x,d) - f_x(x,c) \, dx
\]
\[
= \int_a^b f_x(x,d) \, dx - \int_a^b f_x(x,c) \, dx
\]
(by the FTC) \( \rightarrow \) \[
f(b,d) - f(a,d) - (f(b,c) - f(a,c))
\]
\[
= f(b,d) - f(a,d) - f(b,c) + f(a,c).
\]

**Exercise.** Prove that

\[
\int \int_R f_{yx} \, dA = f(b,d) - f(a,d) - f(b,c) + f(a,c).
\]

This will complete the proof of the lemma. □

**Theorem.** If \( f_{xy} \) and \( f_{yx} \) are continuous, then they are equal.

**Proof** by contradiction. Suppose they are not identically equal. Then at some point \((a,b)\), they differ; say

\[
f_{xy}(a,b) - f_{yx}(a,b) = h > 0.
\]

Then by continuity, there is some small \( \Delta x \times \Delta y \) rectangle, centered at \((a,b)\), on which

\[
f_{xy}(x,y) - f_{yx}(x,y) \geq \frac{h}{2},
\]

so that

\[
\int \int_R f_{xy} - f_{yx} \, dA \geq \int \int_R \frac{h}{2} \, dA = \frac{h}{2}(\Delta x)(\Delta y) > 0.
\]

But this contradicts the lemma: since \( f_{xy} \) and \( f_{yx} \) have equal integrals over the rectangle, necessarily also

\[
\int \int_R f_{xy} - f_{yx} \, dA = \int \int_R f_{xy} \, dA - \int \int_R f_{yx} \, dA = 0. \quad \blacksquare
\]