An Introduction to the Complex Numbers

I. Heuristic beginnings.

In many mathematical contexts, it would be convenient to have a number system that includes a square root of \((-1)\). To achieve this end, mathematicians began\(^1\) by just defining such a number into existence and seeing how far they could get with it. I will follow that approach in this section. Let \(i\) be—well, something—with the property that

\[ i^2 = -1. \]

Obviously, this \(i\), whatever it is, will not be an ordinary real number, because no real number has this property. A number system containing this \(i\) would have to allow addition, subtraction, multiplication and division of its numbers; so it would also have to include the products \(bi\) for each real \(b\), and then the sums \(a + bi\) for real numbers \(a\) and \(b\). After one allows just these particular products and sums, two things turn out to happen:

**Theorem 1.** [i], There are no repeats among the numbers \(\{a + bi : a, b \text{ real}\}\): in other words,

\[ a + bi = c + di \implies a = c \text{ and } b = d. \]

[ii], The set \(\{a + bi : a, b \text{ real}\}\) is closed under addition, under subtraction, under multiplication, and under division by elements other than zero.

**Proof.** [i]: Suppose \(a + bi = c + di\) for real numbers \(a, b, c,\) and \(d\). Then by algebra,

\[ a - c = (d - b)i. \] \[ (*) \]

If \(d - b \neq 0\), then we could solve for \(i\):

\[ i = \frac{a - c}{d - b}, \]

which would make \(i\) a real number, contradiction. So \((d - b) = 0\), and hence, in (*) \((a - c) = 0\) as well.

[ii]: Using ordinary algebra and the equation \(i^2 = (-1)\), it is easy to verify that

\[ (a + bi) \pm (c + di) = a \pm c + (b \pm d)i; \]
\[ (a + bi) \cdot (c + di) = ac - bd + (bc + ad)i; \]

and, if \(c + di \neq 0\),\(^2\)

\[ \frac{1}{c + di} = \frac{c}{c^2 + d^2} - \left( \frac{d}{c^2 + d^2} \right) i. \]

II. The Formal Definition.

The section above shows a little about what this number system with an “\(i\)” (called the system of complex numbers) must be like, provided that it exists at all. The computations in the first section do not guarantee this; after all, seat-of-the-pants computations with a new symbol are not enough to guarantee that there really is a consistent complex number system where everything works. We need and still lack a precise definition of the complex number system as a particular algebraic system: a clearly defined set with a clearly defined addition and a clearly defined multiplication. We will accomplish this by building on what we already have: the real numbers \((\mathbb{R})\). Mathematicians have a method for building \(\mathbb{Z}\) from \(\mathbb{N}\), another for building \(\mathbb{Q}\) from \(\mathbb{Z}\), and a third one for building \(\mathbb{R}\) from \(\mathbb{Q}\); thus, number systems \(\mathbb{Z}, \mathbb{Q},\) and \(\mathbb{R}\) are not in doubt. What

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1 Well, it is plausible that they may have done this, but I don’t know for a fact that they actually did. I have done no research into the actual history of the development of this subject.

2 I explain below where this mysterious formula comes from.
is needed here is the fourth step: the method mathematicians have for constructing of the set of complex numbers \( \mathbb{C} \), starting from \( \mathbb{R} \).

The heuristics above indicate how to do this. The fact that

\[
a + bi = c + di \iff a = c \text{ and } b = d
\]

is structurally identical to the fact that for points in the plane

\[
(a, b) = (c + d) \iff a = c \text{ and } b = d,
\]

which suggests identifying each complex number “\( a + bi \)” as the point \((a, b)\) in the plane; that is, as a set, \( \mathbb{C} \) is just defined to be \( \mathbb{R} \times \mathbb{R} \). As for the operations: the computation

\[
(a + bi) \pm (c + di) = a \pm c + (b \pm d)i
\]

suggests defining

\[
(a, b) + (c, d) := (a + c, b + d), \tag{1}
\]

and the computation

\[
(a + bi) \cdot (c + di) = ac - bd + (bc + ad)i
\]

suggests defining

\[
(a, b) \cdot (c, d) := (ac - bd, bc + ad). \tag{2}
\]

This is in fact what one does. The system of complex numbers is realized as the set of points in the plane, with the addition and multiplication defined in (1) and (2) above. Then, formal versions of the computations in part [ii] of Theorem 1 (and a few more computations to demonstrate things like commutativity and associativity) prove the following theorem:

**Theorem 2.** [i]: The set \( \mathbb{C} = \mathbb{R} \times \mathbb{R} \), together with the addition defined in (1) is an abelian group; [ii]: The set \( \mathbb{R} \times \mathbb{R} - \{(0,0)\} \), together with and multiplication defined in (2), is an abelian group; [iii]: The multiplication distributes over the addition. \( \blacksquare \)

(Such an algebraic structure is called a field.)

**Outline of Proof.** As an additive group, \( \mathbb{C} \) is the Abelian group \( \mathbb{R} \oplus \mathbb{R} \). By routine computations, one easily checks that multiplication is commutative and associative; that multiplication distributes over addition; that the multiplicative identity is \((1,0)\); and that the multiplicative inverse of \((a,b) \neq (0,0)\) is

\[
\left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right).
\]

\( \blacksquare \)

**III. Removing the quotes from “\( a + bi \)”**.

We now have the point \((a, b)\) representing the complex number “\( a + bi \)”, but the but the plus sign and the implied multiplication in the expression “\( a + bi \)” are not instances of the addition and multiplication defined on \( \mathbb{R} \times \mathbb{R} \) in (1) and (2). It will be very convenient to rectify this situation, and we can do so. First: it is easy to see that the set of points \( \{(a,0): a \text{ in } \mathbb{R}\}\)—that is, the \( x \)-axis—behaves exactly the way the set of real numbers behaves (exercise 1 below). So we identify \( \mathbb{R} \) with the \( x \)-axis of \( \mathbb{C} \) (henceforth: the “real axis”) and simply write “\( a \)” for the complex number \((a,0)\). Second: because \((0,1)\) represents “\( 0 + 1i \)” we define

\[
i := (0,1);
\]

that is, from now on the letter \( i \) stands for the point \((0,1)\). As you would hope, the square of this complex number is indeed \((-1)\) (exercise 2). Finally, having identified entities “\( a \)” “\( b \)” and “\( i \)” as points in the plane,
we can verify that \((a, b)\) does indeed equal \(a + b \cdot i\) (exercise 3), so that we may treat \(a + bi\) as an arithmetic expression and not just a name for the point \((a, b)\).

**Exercise 1:** Show, for any real numbers \(a\) and \(b\):

[i], \((a, 0) + (b, 0) = (a + b, 0)\);

[ii], \((a, 0) \cdot (b, 0) = (ab, 0)\).

**Exercise 2:** Show that \((0, 1) \cdot (0, 1) = (−1, 0)\);

that is, that \(i^2 = (−1, 0)\).

**Exercise 3:** Show for any \((a, b)\) in \(C\), that

\((a, b) = (a, 0) + (b, 0) \cdot (0, 1)\);

that is, that \((a, b) = a + b \cdot i\).

**IV. Modulus and Conjugate.**

For each complex number \(z = a + bi\), one defines the **modulus** (or **norm**) of \(z\) to be the number

\[
|z| := \sqrt{a^2 + b^2}
\]

and the **complex conjugate** of \(z\) to be the number

\[
\overline{z} := a - bi.
\]

Obviously, \(|z|\) is the distance from \(z\) to the origin, and \(\overline{z}\) is the reflection of \(z\) in the real axis. Obviously also, \(\overline{z} = z\) if and only if \(z\) is a real number (that is, if and only if \(b = 0\)).

The following facts are proved by follow-your-nose, straightforward computations.

**Exercise 4:** Prove:

[a], \(\overline{z + w} = \overline{z} + \overline{w}\).

[b], \(\overline{zw} = \overline{z} \overline{w}\).

[c], \(z \overline{z} = |z|^2\).

I can now explain the origin of the mysterious formula for \(\frac{1}{c + di}\) (see p.1). Let \(z = (c, d) \neq (0, 0)\) in \(C\). Then

\[
1 = \frac{|z|^2}{|z|^2}
\]

exercise 4[c] \(\rightarrow \frac{z \overline{z}}{|z|^2} = \frac{z \overline{z}}{|z|^2}\),

\[
\text{regroup} \rightarrow = z \left(\frac{\overline{z}}{|z|^2}\right).
\]

Therefore, the multiplicative inverse of \(z\) is the thing in the parentheses:

\[
\frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{c}{|z|^2} - \frac{di}{|z|^2}.
\]

I am also in a position to demonstrate one of the two key facts that explain how to understand complex multiplication geometrically.

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3 \(|z|\) is a nonnegative real number, but keep in mind that we have identified it with the complex number \((|z|, 0)\), since \(\mathbb{R}\) has been identified with the real axis in \(C\).
**Theorem 3.** For any complex numbers $z$ and $w$,

$$|zw| = |z| \cdot |w|.$$ 

**Proof.** I will prove that both of these numbers have the same square; since they’re both nonnegative real numbers, it will follow that they’re the same number.

$$|zw|^2 =$$

exercise 4[c] $\rightarrow = z\bar{w} \bar{z}w$

exercise 4[b] $\rightarrow = z\bar{w} \bar{z} \bar{w}$

exercise 4[c] $\rightarrow = |z|^2 |w|^2$. 

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**V. Two Additional Important Facts about $\mathbb{C}$.**

I have now proved everything I want to prove to you about $\mathbb{C}$, but I want to tell you two more facts about $\mathbb{C}$ without proving what I say. The first fact is the second key ingredient in the geometric picture of multiplication. As noted above, Theorem 3 tells you that when you multiply complex numbers $z$ and $w$, you multiply their moduli; in other words, $zw$ will lie somewhere on the circle of radius $|z||w|$ around the origin. The other geometric fact about multiplication tells you exactly where on this circle $zw$ will lie. It says that you add the angles that $z$ and $w$ (as radii) make with the positive real axis to find the angle that $zw$ makes with the positive real axis. So, for example: suppose $|z| = 2$, and $z$ makes a $30^\circ$ angle with the positive $x$–axis; and suppose and $|w| = 3$, and $w$ makes a $45^\circ$ angle with the positive $x$–axis. Then $zw$ is on the circle of radius 6 centered at the origin; and the radius from $zw$ to the origin will make that makes an angle of $75^\circ$ with the positive $x$–axis.

The second fact is much deeper. It addresses the question: will we ever want to extend the set of numbers even further, beyond $\mathbb{C}$? $\mathbb{C}$ is the result when we extend $\mathbb{R}$ by throwing in a solution to the equation $x^2 + 1 = 0$; will we ever extend $\mathbb{C}$ to some larger system by throwing in a solution to some other polynomial equation? The answer to this question turns out to be no: all solutions to all polynomial equations are already present in $\mathbb{C}$. The precise theorem is:

**Theorem 4 (Fundamental Theorem of Algebra).** Every polynomial of degree $n \geq 1$ with complex coefficients will have (counting multiple roots) exactly $n$ solutions in $\mathbb{C}$.

Thus, for example, you will not need to extend $\mathbb{C}$ to find the fourth roots of $(-1)$ (the solutions to $x^4 + 1 = 0$). They’re already there.

**Exercise 5:** Using the geometric interpretation of multiplication, find a complex number $z$ such that $z^4 = -1$.

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\[4\text{ Actually, there are four of them. Can you find them all??}\]