Partial Fractions and Integrating Rational Functions

The term <i>partial fractions</i> refers to an algebraic technique which is usually included in a presentation of methods for integrating functions of the form \( \frac{f(x)}{g(x)} \), where \( f(x) \) and \( g(x) \) are polynomials. It is a fact that all such functions (which are called <i>rational functions</i>) can be integrated, at least in principle. In practice, some of the techniques you need are quite laborious, and others may sometimes be impossible to apply. I am going to ask you to learn to apply these techniques only in a few special cases, which I will discuss in class. Here, I want to give you an idea of the big picture, which is easy to grasp in outline.

**Step I.** If the degree of \( f(x) \) is larger than or equal to the degree of \( g(x) \), you can divide \( f(x) \) by \( g(x) \), getting quotient \( q(x) \) and remainder \( r(x) \), where \( q(x) \) is a polynomial, and \( r(x) \) is either zero or a polynomial whose degree less than the degree of \( g(x) \). This will allow you to rewrite the integrand as \[
\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}.
\]
Since \( \int q(x) \, dx \) is trivial to compute, this reduces the original integration problem to that of finding \( \int \frac{r(x)}{g(x)} \, dx \) (where the degree of the numerator is smaller than that of the denominator). In class, I will discuss this step in detail; I want you to know this part well.

**Step II.** Factor \( g(x) \) as far down as possible, using real numbers (both rational and irrational) for coefficients. It can be proved that \( g(x) \) can in principle be factored into factors of degree at most 2. For example, here is a typical complete factorization:
\[
g(x) = 3(x - 3)^3(x - \pi)(x^2 + \sqrt{2}x + 4)^3.
\]
Unfortunately, these factors are sometimes impossible to find in practice. I will bother you with only very small cases in class.

**Step III.** Using the factorization from Step II, it is algebraically possible to write \( \frac{r(x)}{g(x)} \) as a sum of simpler fractions, using powers of the factors of \( g(x) \) for denominators. Here is an example: if \[
g(x) = (x - 5)^3(x + 3)(x^2 + 2x + 3)^2,
\]
then you can find (unique) constants so that
\[
\frac{r(x)}{g(x)} = \frac{c_1}{x - 5} + \frac{c_2}{(x - 5)^2} + \frac{c_3}{(x - 5)^3} + \frac{c_4}{x + 3} + \frac{c_5}{x^2 + 2x + 3} + \frac{c_6}{(x^2 + 2x + 3)^2}.
\]
This is called the <i>partial fraction decomposition</i> of \( \frac{r(x)}{g(x)} \). Finding the constants is routine but laborious; I will discuss only very simple examples in class.

**Step IV.** All of the terms you get in (1) can in fact be integrated. In class, I will discuss how to integrate some of them but not all.