Root Test. Suppose that the following limit exists:

\[ L = \lim_{k \to \infty} \sqrt[k]{|a_k|}. \]

Then

\[
\begin{align*}
[a]: & \quad L < 1 \implies \sum_{k=0}^{\infty} a_k \text{ converges absolutely;} \\
[b]: & \quad L > 1 \implies \sum_{k=0}^{\infty} a_k \text{ diverges;} \\
[c]: & \quad \text{if } L = 1, \quad \sum_{k=0}^{\infty} a_k \text{ might converge absolutely, converge conditionally, or diverge.}
\end{align*}
\]

Proof. [a]. Assume \( L < 1 \), and choose any number \( r \) that is strictly between \( L \) and 1: that is, \( L < r < 1 \). Now \( \lim_{k \to \infty} \sqrt[k]{|a_k|} = L < r \), so for all \( k \) large enough—say, for all \( k \geq K_0 \)—it must be true that

\[ \sqrt[k]{|a_k|} \leq r, \]

which implies that

\[ |a_k| \leq r^k. \]

Next: since \( r < 1 \), the geometric series

\[ 
\sum_{k=K_0}^{\infty} r^k = r^{K_0} + r^{K_0} \cdot r + r^{K_0} \cdot r^2 + \cdots \text{ converges to } \frac{r^{K_0}}{1-r};
\]

so by the PVCT, \( \sum_{k=K_0}^{\infty} |a_k| \) also converges. Finally, this means that \( \sum_{k=K_0}^{\infty} a_k \) converges absolutely.

[b]: If \( \lim_{k \to \infty} \sqrt[k]{|a_k|} = L > 1 \), then for all \( k \) large enough—say, for all \( k \geq K_0 \)—it must be true that

\[ \sqrt[k]{|a_k|} \geq 1, \]

which implies that

\[ |a_k| \geq 1^k = 1. \]

This makes it impossible for \( \lim_{k \to \infty} a_k \) to equal zero, so by the Test for Divergence, \( \sum_{k=K_0}^{\infty} a_k \) diverges.

[c]: To prove the third assertion, it is sufficient to exhibit three series—one absolutely convergent, the second conditionally convergent, and the third divergent—for which \( L = 1 \). Three series that work are

\[ \sum_{k=1}^{\infty} \frac{1}{k^2}, \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}, \text{ and } \sum_{k=1}^{\infty} \frac{1}{k}. \]

The first of these converges absolutely (\( p \text{-series, } p = 2 \)), the third one diverges (\( p \text{-series, } p = 1 \)), the second one converges conditionally—it converges by the Alternating Series Test, and convergence is conditional because \( \sum_{k=1}^{\infty} \left| (-1)^{k+1} \frac{1}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k} \) diverges. Moreover, routine L’Hospital’s Rule computations show that \( L = 1 \) for all of these series.

Extra credit: Fifteen points extra credit on the quiz total of anyone who computes \( L \) for these three series.