Dealing With Proofs

1 Introduction.

A proof is an organized logical argument whose purpose is to establish the truth of a mathematical statement; to mathematicians, proofs are essential for advancing knowledge and for communicating it. As a consequence, students who specialize in math learn to deal with proofs early on, and I wrote this handout as a guide for such students to use when they are encountering proofs for the first time; Of course, students’ exposure to proofs in MAT 130 is much less extensive than it is in specialized courses. Nevertheless, I think you will find this handout useful.

2 Reading Proofs.

1. Make sure you understand the terms. In mathematics, there are many specialized terms (words with reserved meanings). Some of them are standard (for example, “congruent mod 5”), while others may have their reserved meanings only in a very limited context (for example, the vector $p$ in theorems 3.3 and 3.4 of the Cauchy-Schwarz handout). In order to understand a proof, you must know which words are the reserved words and what every reserved word means.

2. Read proofs more than once. When you first read a difficult proof, your goal should be to understand its logical structure. You may not know how anybody thought the proof up, or you may think that the theorem being proved is pointless and not worth proving. Often, understanding the point of a theorem or the strategy behind a proof will come later, on a second or third reading. In the same way, you should not expect to understand every line of every proof discussed in class, but the ideas should become clear as you reread your class notes.

3. Write things down. If you really want to master the material—to make it yours—you should write notes—your insights and questions—meant for your eyes only. The first time you study the material, paraphrase the main points and write down any questions you may have; you may find yourself answering your own questions the second time through, when your understanding has grown clearer. (Such notes will be of immense help, by the way, when you are studying for an exam.)

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1. These will always have been formally defined, frequently the word “definition” marks the spot where this is being done.

2. When a proof employs truly original ideas, you may never know how the prover ever thought of them. Nonetheless, now that the ideas have been written down, you can make them yours. In mathematics, more than in any other field, the ideas of the giants can be absorbed, understood, and used by the rest of us.

3. As mentioned in the syllabus: this would make a good inclusion for your journal.
3 Discovering Proofs.

There are no universal rules here; figuring out how to prove an assertion is an open-ended process. Here are a few things to keep in mind, though:

**Getting Started.** This will sometimes be the hardest part; once you begin, you may find that you can continue. Here are some tips that might help you get started.

- The first step in discovering why something is true is understanding what it says. It can be useful to “unwrap” a definition or to see what a known theorem says when applied to the specific case at hand.
- Playing with examples can sometimes illuminate what is going on.
- You can prove a statement true by showing that if it were false, a contradiction would result.
- If you are having great difficulty proving a statement or even believing it, maybe it is false. See if you can show that it is not true.

**In the Middle.** Suppose you have made some progress but do not see how to advance any further. Here are some things to keep in mind.

- You may need to “focus on the next ten yards.” In order to score a touchdown, trying just to move the ball ten yards for the next first down is sometimes a wiser strategy than throwing a long bomb. Similar wisdom sometimes will move a proof along: it is sometimes better to ignore the final objective and focus instead on what you know at the moment. What you have right in front of you may well suggest a next step.
- Sometimes, though, it helps to take another look at the final objective (i.e., what you are trying to prove). Reminding yourself of where you want to go will sometimes guide your thinking.
- Check to see whether you have used all of the assumptions; an as-yet-unused assumption may point the way forward.
- Algebra is often helpful.
- Recasting a statement into a different but logically equivalent form can sometimes help. In particular, it may be easier to prove the contrapositive of an “If \(X\), then \(Y\)” statement.

**Ways to Avoid Getting Derailed.**

- NEVER assume your conclusion.
- Be precise. Vague thinking allows mistakes to slip in, and vague language can cloud your thinking.
- Avoid vague pronouns. A common example of vagueness is *its* and *that* which could refer to two or more different things: losing track of what a pronoun refers to can completely invalidate a proof. The easiest way to avoid this problem is simply to avoid pronouns wherever possible.
- Things that are not necessarily the same MUST NOT be represented by the same symbol. (This is the mathematical version of the vague-pronoun error.)

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4 temporarily!

5 Throwing a long bomb is not always the wrong strategy! Keep your options open!

6 Here is an example. Let \(n\) and \(k\) be positive integers, and let \(p\) be a prime. The statement “If \(p\) divides \(nk\), then either \(p\) divides \(n\) or \(p\) divides \(k\)” can be proved by showing: “If \(p\) divides \(nk\) and \(p\) does not divide \(n\), then \(p\) must divide \(k\).”

7 Even if a vague pronoun does not lead to an error, it is still bad writing.
4 Writing Proofs.

Say you have successfully tackled a proof problem: you see why the assertion must be true. You now face the task of writing up your solution. Even if you have solved the problem completely and you understand your solution thoroughly, it is still not easy to write a clear, coherent account of your proof. This job demands thought, discipline, effort, and—usually—revisions. The following tips and guidelines should make your efforts more effective and improve your final draft.

1. **Make sure you have understand everything you are writing.** Don’t write a solution until you have it in clear focus. Clarify any fuzzy links and plug any gaps in your chain of reasoning. You will find that clear understanding will actually solve many (although not all!) of the writing problems in advance.

2. **If your solution is incomplete...** So, what if you got partway to the goal but got stuck? Or what if, say, you do not see how to plug a gap, or you need to add an assumption to make your proof go through? In this case, you should **write what you have and explain how it falls short.** Doing so improves your writeup: it tells the reader that you are aware of the gaps—that you yourself know what you have proved and what you have not proved.

3. **Have an outline of your entire proof in mind as you write.** Keep track of what you are assuming, what you are proving, and where you are in your proof. You should be especially careful if your proof is an argument by contradiction, or if you are arguing the contrapositive of a given assertion.

4. **Include enough words...** Your attempt to communicate your ideas to the reader will often stand or fall by how much “connective tissue” you include. I am talking about sentences like these:

   - “The problem as stated is equivalent to statement XXXX. I will prove XXXX.”
   - [If you plan to prove \(X \implies Y\) by proving the contrapositive \(\neg(Y) \implies \neg(X)\): “Suppose, on the contrary, that \(Y\) is false.”]
   - “I must show that XXXX and YYYY together imply ZZZZ. I will do this by showing how to get a contradiction by assuming XXXX and YYYY and not-ZZZZ.”
   - “Such a function cannot possibly exist: assuming that you have any such function leads to the following contradiction.”

   Such sentences serve as logical guide-posts for the reader: they set out the large-scale structure of the argument—the main ideas that made your proof work.

5. **...but don’t include extra words.** Don’t bury your argument in irrelevant details. In the process of finding a proof, for example:

   - you may start by working specific examples to get initial insights into what is going on;
   - you may first prove a special case, using extra assumptions, that you later get rid of;
   - or you may discover side facts that you don’t wind up using in the final argument.

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8 Part of your job in this course will be to hone and strengthen your proof-writing skills, and part of my job will be to assist you.
9 or however much you have of the solution
Exclude all such extraneous matter from your final draft. Such irrelevancies are like the scaffolding that used to construct a building: after the building is finished, the scaffolding is no longer needed, and it hides the completed structure.

6. **Be clear and precise.** Imprecise mathematical prose often indicates lingering fuzziness in the mathematics itself, and you will find that clarifying your explanation will often clarify your thinking. You can make your prose clearer by writing with the following goals in mind. (I’m sure there are others I haven’t thought of!)

   - **Avoid vague pronouns** (see p. 3). If you do use pronouns, make sure that each of them has an unambiguous referent.
   - **Define any new terms** that you introduce. When you introduce a variable, state precisely what it represents.
   - **Be careful in your use of logic.** Make all quantifiers (“for some” and “for all” phrases) explicit; don’t confuse an implication with its converse; be especially careful if you must express the negation of a complicated logical statement.

7. **Choose your notation carefully.** Well-chosen notation streamlines exposition and sometimes actually simplifies an argument, making it easier to follow. Poorly chosen or imprecise notation, on the other hand, makes it hard for the reader to follow your argument and may even wind up derailing your proof.

8. **Read your proof aloud.** After you have written your proof down, read it aloud. This will help you gauge how it will sound (and look) to others. You may discover an error, a gap, or—even if your solution is correct—a place where your explanation is unclear.

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10For example, here is a puzzle: are my grandmothers’ great-grandmothers the same people as my great-grandmothers’ grandmothers? Good notation is of immense help for this one.