The Completeness of the Propositional Calculus

Hilbert’s program for the Propositional Calculus met with complete success: the “Propositional Calculus robot” functions perfectly. We showed in class on September 22 that $L$ is consistent—that there is no wf $A$ such that both $\vdash L \neg A$ and $\vdash L A$. The purpose of this handout is to show that $L$ is also complete, which means roughly that $L$ can deduce every wf we’d like it to be able to. (Below, I offer two possible precise meanings that one might want to give to this term.) The main result will be established in three lemmas; but before getting to those, I need to establish a few technical details about $L$.

I. Five Formal Deductions and a Deduction Rule.

1: $\vdash L \neg \neg \neg B \rightarrow \neg B$.

1. $\neg \neg \neg B \rightarrow \neg B$ Deduction Theorem applied to instance of Hamilton #2b (p 36)
2. $(\neg \neg \neg B \rightarrow \neg B) \rightarrow (B \rightarrow \neg \neg B)$ L3
3. $B \rightarrow \neg \neg B$ MP(1,2)

2: $\vdash L (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$.

This is Hamilton #3b (p 36).

3: $\vdash L (A \rightarrow \neg A) \rightarrow \neg A$.

1. $(A \rightarrow \neg A) \rightarrow \neg A$ Instance of Hamilton, Prop. 2.11b (p 35)
2. $(A \rightarrow \neg A) \rightarrow (\neg \neg A \rightarrow \neg A)$ Instance of #2 above
3. $(A \rightarrow \neg A) \rightarrow \neg A$ HS(2,1)

4: $\vdash L A \rightarrow ((A \rightarrow B) \rightarrow B)$.

This can be obtained by applying the Deduction Theorem twice to the obvious three-line deduction of $\{A, A \rightarrow B\} \vdash L B$.

5: $\vdash L \sim (A \rightarrow B)$.

1. $A \rightarrow ((A \rightarrow B) \rightarrow B)$ #4 above
2. $A$ Assumption
3. $(A \rightarrow B) \rightarrow B$ MP(2,1)
4. $((A \rightarrow B) \rightarrow B) \rightarrow (\sim B \rightarrow \sim (A \rightarrow B))$ Instance of #2 above
5. $\sim B \rightarrow \sim (A \rightarrow B)$ MP(3,4)
6. $\sim B$ Assumption
7. $\sim (A \rightarrow B)$ MP(6,5)

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1 The methods used in this handout all pass Intuitionistic muster. This handout is based on the proof that appears in S. C. Kleene, *Introduction to Metamathematics*, Van Nostrand (1952).
Deduction Rule. If

\[
\begin{align*}
\{\alpha\} \Gamma \cup \{A\} \vdash_L C \\
\{\beta\} \Gamma \cup \{B\} \vdash_L C,
\end{align*}
\]

then

\[
\Gamma \cup \{A \lor B\} \vdash_L C.
\]

(Recall that “\(A \lor B\)” is a name for \((\sim A) \to B\).)

1. \(A \to C\) Deduction Theorem applied to \((\alpha)\)
2. \(B \to C\) Deduction Theorem applied to \((\beta)\)
3. \((\sim A) \to C\) Assumption
4. \((\sim A) \to C\) HS(3,2)
5. \((A \to C) \to (\sim C \to \sim A)\) Instance of \#2 above
6. \(\sim C \to \sim A\) MP(1,5)
7. \(\sim C \to C\) HS(6,4)
8. \((\sim C \to C) \to C\) Instance of Hamilton, Prop. 2.11b (p35)
9. \(C\) MP(7,8)

II. The Main Lemmas.

**Proposition C1.** Let \(A\) be a proposition in distinct proposition letters \(\{p_1, \ldots, p_m\}\), and focus on any one line (line \(j\), say) of the \(2^n\) lines of the truth table of \(A\). Using this line \(j\), set

\[
q_i := \begin{cases} 
  p_i, & \text{if } p_i \text{ is true in line } j; \\
  \sim p_i, & \text{if } p_i \text{ is false in line } j.
\end{cases}
\]

Then: [a], if \(A\) is true in line \(j\), \(\{q_1, \ldots, q_m\} \vdash_L A\); and [b], if \(A\) is false in line \(j\), \(\{q_1, \ldots, q_m\} \vdash_L \sim A\).

**Proof** by induction by the number \(n\) of occurrences of “\(\sim\)” and “\(\to\)” in \(A\).

**Basis** \((n = 0)\). Then \(A = p_i\). If \(p_i\) is true in line \(j\), then \(q_i = p_i\), so \(\{q_1, \ldots, q_m\} \vdash_L A\) by the one-line deduction.

1. \(p_i\) assumption

On the other hand, if \(p_i\) is false in line \(j\), then \(q_i = \sim p_i\), so \(\{q_1, \ldots, q_m\} \vdash_L \sim A\) by the one-line deduction.

1. \(\sim p_i\) assumption

**Inductive step.** Assume both [a] and [b] hold for all propositions in \(\{p_1, \ldots, p_m\}\) with \(0 \leq k \leq (n - 1)\) occurrences of “\(\sim\)” and “\(\to\)”, and let \(A\) have \(n\) occurrences of “\(\sim\)” and “\(\to\)”.

**Case I:** \(A\) is \(\sim B\) for some proposition \(B\).

**Case I(\(\alpha\))**: If \(A\) is true (so \(B\) is false), then by induction \(\{q_1, \ldots, q_m\} \vdash_L \sim B\); that is, \(\{q_1, \ldots, q_m\} \vdash_L \sim A\).

**Case I(\(\beta\))**: If \(A\) is false (so \(B\) is true), then by induction \(\{q_1, \ldots, q_m\} \vdash_L B\). Add to this deduction:

\[
\begin{align*}
\ell &. B \\
\ell + 1. & \sim B \to \sim B \\
\ell + 2. & \sim B \to B \\
\end{align*}
\]

Formal deduction \#1 of section I

\(\ell\), \(\ell + 1\), MP(\(\ell, \ell + 1\)),

and this is what we want, because \(\sim B\) is \(\sim A\).

**Case II:** \(A\) is \(B \to C\) for some propositions \(B\) and \(C\).

**Case II(\(\alpha\))**: Say \(B\) is false (so \(A\) is true). By induction, we have \(\{q_1, \ldots, q_m\} \vdash_L \sim B\). Add to this deduction:

\[
\begin{align*}
\ell &. \sim B \\
\ell + 1. & \sim B \to (B \to C) \\
\ell + 2. & B \to C \\
\end{align*}
\]

Hamilton, Prop. 2.11a(p 35)

\(\ell\), \(\ell + 1\), MP(\(\ell, \ell + 1\))
Case II(β): Say \( C \) is true (so \( A \) is true). By induction, we have \( \{q_1, \ldots, q_m\} \vdash C \). Add to this deduction:

\[
\begin{align*}
\ell & \quad C \\
\ell + 1 & \quad C \rightarrow (B \rightarrow C) \\
\ell + 2 & \quad B \rightarrow C \\
\end{align*}
\]

Proof:\( L1 \)

\( \text{MP}(\ell, \ell + 1) \)

Case II(γ): Say \( B \) is true and \( C \) is false (so \( A \) is false). By induction, we have both \( \{q_1, \ldots, q_m\} \vdash \sim B \) and \( \{q_1, \ldots, q_m\} \vdash \sim C \). Concatenate these deductions, and add the following:

\[
\begin{align*}
k & \quad B \\
\ell & \quad \sim C \\
\ell + 7 & \quad \sim (B \rightarrow C),
\end{align*}
\]

and this is what we want, because \( \sim (B \rightarrow C) \) is \( \sim A \). (Induction is complete.)  

Proposition C2. Again, let \( A \) be a proposition in distinct proposition letters \( \{p_1, \ldots, p_m\} \). If \( A \) is a tautology, then \( \{p_1 \lor (\sim p_1), p_2 \lor (\sim p_2), \ldots, p_m \lor (\sim p_m)\} \vdash A \).

Proof: is by \( (2^{m-1} + 2^{m-2} + 2^{m-3} + \cdots + 2 + 1) \) applications of the deduction rule established in section I above.\(^2\) (Let me call this “Deduction Rule (*).”) The argument, as you will see, has the same structure as the chart for an elimination tournament.

First: Because \( A \) is a tautology, Proposition C1 above gives \( 2^m \) deductions of \( A \) from \( 2^m \) different sequences of assumptions, namely:

\[
\begin{align*}
(1) & \quad \{p_1, \ldots, p_m-1, p_m\} \vdash A \\
(2) & \quad \{p_1, \ldots, p_m-1, \sim p_m\} \vdash A \\
(3) & \quad \{p_1, \ldots, \sim p_m-1, p_m\} \vdash A \\
(4) & \quad \{p_1, \ldots, \sim p_m-1, \sim p_m\} \vdash A \\
\vdots & \quad \vdots \\
(2^m) & \quad \{\sim p_1, \ldots, \sim p_m-1, \sim p_m\} \vdash A
\end{align*}
\]

Now apply Rule (*) to each of the \( 2^{m-1} \) adjacent pairs of lines above, using in each case the assumptions \( p_m \) and \( \sim p_m \):

\[
\begin{align*}
(1') & \quad \{p_1, \ldots, p_m-1, p_m \lor \sim p_m\} \vdash A \\
(2') & \quad \{p_1, \ldots, \sim p_m-1, p_m \lor \sim p_m\} \vdash A \\
\vdots & \quad \vdots \\
(2^{(m-1)}) & \quad \{\sim p_1, \ldots, \sim p_m-1, p_m \lor \sim p_m\} \vdash A
\end{align*}
\]

Now apply Rule (*) to each of the \( 2^{m-2} \) adjacent pairs of lines above, using in each case the assumptions the \( p_{m-1} \) and \( \sim p_{m-1} \):

\[
\begin{align*}
(1'') & \quad \{p_1, \ldots, p_m-1 \lor \sim p_m-1, p_m \lor \sim p_m\} \vdash A \\
\vdots & \quad \vdots \\
(2^{2m-2}) & \quad \{\sim p_1, \ldots, p_m-1 \lor \sim p_m-1, p_m \lor \sim p_m\} \vdash A
\end{align*}
\]

Continuing in this way, after a total of \( (2^{m-1} + 2^{m-2} + 2^{m-3} + \cdots + 2 + 1) \) applications of (*), you finally

\(^2\) As you will see, the proof has the structure of a “March Madness” bracket.
arrive at one single deduction

\[
\{p_1 \lor \neg p_1, p_2 \lor \neg p_2, \ldots, p_m \lor \neg p_m\} \vdash \mathcal{A}. \quad \blacksquare
\]  

(†)

Proposition C3. Let \( \mathcal{A} \) be a proposition in distinct proposition letters \( \{p_1, \ldots, p_m\} \). If \( \mathcal{A} \) is a tautology, then \( \vdash \mathcal{A} \).

Proof. Exercise 1. (Hint: Tinker with the justification side of deduction (†).)

A tautology, note, is a proposition that expresses a logical truth—that is, one that must be true by the rules of propositional logic alone, regardless what particular statements the proposition letters might represent. The first possible precise meaning of the statement “\( \mathcal{L} \) is complete” is that all such logical truths should be deducible in \( \mathcal{L} \); Proposition C3 says that \( \mathcal{L} \) is complete in this sense. The other possible precise definition of “completeness” is a saturation criterion: \( \mathcal{L} \) would be complete in the second sense if you were unable to adjoin to the axioms of \( \mathcal{L} \) any wfs (unless they are already theorems) without rendering the system inconsistent. As it turns out, \( \mathcal{L} \) is not complete in this sense—nor would we want it to be!—but it does have the property that if you add as a fourth axiom a schema of the same sort that \((\mathcal{L}1), (\mathcal{L}2)\) and \((\mathcal{L}3)\) are, and if the new schema is not already a theorem of \( \mathcal{L} \), then the extended system will necessarily be inconsistent:

Proposition C4. Let \( \mathcal{A} \) be a wf schema that is not a tautology, and let \( \mathcal{L}^+ \) be the system \( \mathcal{L} \) with \( \mathcal{A} \) added as axiom schema \( \mathcal{L}4 \). Then \( \mathcal{L}^+ \) is not consistent.

Proof. Exercise 2. (Hint: Since \( \mathcal{A} \) is not already a theorem, it is not a tautology; start by finding an instance of \( \mathcal{A} \) that is identically false.)