The Structure of the Real Number Line

1 Introduction.

Nineteenth-and early-twentieth-century mathematicians perfected the technique of realizing mathematical structures as set-based axiom systems. Typically, the process involves stipulating some sets, together with some functions and relations on these sets, that satisfy certain properties, which are called “axioms;” you are familiar with one example of such a system from MAT 225, namely the definition of the class of vector spaces. The real number line can also be defined in this way,\(^1\) and the purpose of this handout is to set out the details. This topic is also covered in Chapter 1 of the text, and this handout is intended as a supplement to that discussion.\(^2\)

2 The Definition of \(\mathbb{R}\).

The real number line\(^3\) \(\mathbb{R}\) consists of

- a set (also called “\(\mathbb{R}\)”)
- two elements of \(\mathbb{R}\) called “1” and “0”
- two functions from \(\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\):
  - the **addition** function, \(A(x, y) = x + y\)
  - the **multiplication** function, \(M(x, y) = x \cdot y\)
- a binary relation\(^4\) on \(\mathbb{R}\) written \(x < y\).

In addition: \(\mathbb{R}\) 1, 0, \(A\), \(M\), and \(<\) must satisfy all of the following axioms.

2.1 Addition Axioms.

**Axiom 1+ (commutativity of addition):** For all \(x\) and \(y\) in \(\mathbb{R}\)

\[ x + y = y + x \quad \text{or, equivalently,} \quad A(x, y) = A(y, x). \]

**Axiom 2+ (associativity of addition):** For all \(x\), \(y\), and \(z\) in \(\mathbb{R}\)

\[ x + (y + z) = x + (y + z) \quad \text{or, equivalently,} \quad A(x, A(y, z)) = A(A(x, y), z). \]

**Axiom 4(a) (additive identity):** For all \(x\) in \(\mathbb{R}\)

\[ x + 0 = x. \]

**Axiom 5(a) (additive inverse):** For each \(x\) in \(\mathbb{R}\) there is a number \(y\) in \(\mathbb{R}\) such that \(x + y = 0\). (As we will prove: for each \(x\), there is only one such \(y\), which is denoted \(-x\).)

A set with a binary operation that obeys these four rules is called an **abelian group**, so one can summarize the axioms in §2.1 with the single sentence, “Under the addition operation, \(\mathbb{R}\) is an abelian group.”

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\(^1\)There are other approaches; see §3 on p.4 for a brief discussion of one of these.

\(^2\)I have regrouped the axioms and altered their order of appearance in order to acquaint you with two other important axiom systems: **abelian groups** and **fields**.

\(^3\)Focus for a moment on the word “the” in the definition. This definition is *psychologically* different from the definition of the class of vector spaces in that we expect different vector spaces to behave differently, whereas we hope that the axioms of \(\mathbb{R}\) determine \(\mathbb{R}\) completely. This hope turns out to be justified (see §4 on p.5 for a brief explanation why).

\(^4\)Recall that a binary relation on a set \(A\) can be viewed as a subset of \(A \times A\).
2.2 Multiplication Axioms.

I have stated the axioms in this section only for nonzero real numbers, for two reasons: 1), the axiom statements are simpler for the nonzero reals; and 2), after all of the axioms are in place, we will be able to prove that \( x \cdot 0 = 0 \) for all \( x \) in \( \mathbb{R} \), so that nothing about multiplication by zero needs to be assumed.

Axiom 1 (commutativity of multiplication): For all \( x \) and \( y \) in \( (\mathbb{R}−\{0\}) \),
\[
x \cdot y = y \cdot x \quad \text{or, equivalently,} \quad M(x, y) = M(y, x).
\]

Axiom 2 (associativity of multiplication): For all \( x \), \( y \) and \( z \) in \( (\mathbb{R}−\{0\}) \),
\[
x \cdot (y \cdot z) = (x \cdot y) \cdot z \quad \text{or, equivalently,} \quad M(x, M(y, z)) = M(M(x, y), z).
\]

Axiom 4(b) (multiplicative identity): For all \( x \) in \( (\mathbb{R}−\{0\}) \),
\[
x \cdot 1 = x.
\]  

Axiom 5(b) (multiplicative inverse): For each \( x \) in \( (\mathbb{R}−\{0\}) \), there is a number \( y \) in \( (\mathbb{R}−\{0\}) \) such that \( x \cdot y = 1 \). (As we will prove: for each \( x \) in \( (\mathbb{R}−\{0\}) \), there is only one such \( y \) in \( (\mathbb{R}−\{0\}) \). This \( y \) is denoted \( x^{-1} \) or \( \frac{1}{x} \).)

One can summarize the axioms in §2.2 with the single sentence, “under the multiplication operation, \( (\mathbb{R}−\{0\}) \) is an abelian group.”

2.3 Order Axioms.

Axiom 6. For all \( x \) and \( y \) in \( \mathbb{R} \), exactly one of \( \{ x < y \}, \{ y < x \}, \{ x = y \} \) is true.

Axiom 7. For all \( x \), \( y \) and \( z \) in \( \mathbb{R} \): if \( x < y \) and \( y < z \), then \( x < z \).

2.4 Axioms Relating Addition, Multiplication and Order.

Axiom 3 (Addition/Multiplication): For all \( x \), \( y \) and \( z \) in \( \mathbb{R} \),
\[
x \cdot (y + z) = (x \cdot y) + (x \cdot z).
\]

Axiom 4(c) (Addition/Multiplication): \( 1 \neq 0 \).

Any set with two operations in which
- all of the axioms in §2.1,
- all the axioms in §2.2,
- Axiom 3, and
- Axiom 4(c)

are all valid is called a field.

Axiom 8 (Addition/Order): For all \( x \) and \( z \) in \( \mathbb{R} \): if \( x < y \), then \( x + z < y + z \).

Axiom 9 (Multiplication/Order): For all \( x \), \( y \) and \( z \) in \( \mathbb{R} \): if \( x < y \) and \( z \) is positive,\(^6\) then \( x \cdot z < y \cdot z \).

A set with an addition, a multiplication and an order that satisfies all of the axioms in §§2.1–2.4 is called an ordered field, and there are gazillions of them! All of you are familiar with two of them: \( \mathbb{R} \) itself and \( \mathbb{Q} \), the set of rational numbers.\(^7\)

\(^5\)The text includes this property as a clause in Axiom 4(b) rather than setting it off as a separate axiom.

\(^6\)That is, if \( 0 < z \).

\(^7\)Those seniors who took MAT 404 can name many more.
2.5 The Completeness Axiom.

This is the hard axiom, the one that will require a lot of discussion. As you are aware, there exist lengths, such as the example I mentioned in the syllabus\(^8\), that cannot be represented by fractions. These point-size gaps are everywhere. As a set of points on the number line, \(\mathbb{Q}\) is *porous*; if we restrict ourselves to only rational numbers, all of Real Analysis collapses.

The purpose of the Completeness Axiom is to ensure that the set of real numbers has no such gaps. Before I can state the axiom, I need two definitions.

**Definition 1** Let \(S\) be a subset of \(\mathbb{R}\), and let \(b \in \mathbb{R}\).

1. \(b\) is a(n) \begin{align*}
\text{upper bound} & \quad \text{for } S \quad \text{if} \quad \{s \leq b \mid b \leq s\} \quad \text{for all } s \in S.
\end{align*}

2. If \(S \subseteq \mathbb{R}\) has a(n) \begin{align*}
\text{both an upper and a lower bound}
\end{align*}

then \(S\) is said to be \begin{align*}
\text{bounded above}
\end{align*}

\begin{align*}
\text{bounded below}
\end{align*}

\begin{align*}
\text{bounded}
\end{align*}

Does the empty set have any upper or lower bounds?

**Definition 2**

1. Let \(S\) be bounded above. \(b_0 \in \mathbb{R}\) is a **least upper bound** for \(S\) if
   
   (a) \(b_0\) is an upper bound for \(S\), and
   
   (b) if \(b_1\) is any other upper bound for \(S\), then \(b_0 < b_1\).

2. Let \(S\) be bounded below. \(b_0 \in \mathbb{R}\) is a **greatest lower bound** for \(S\) if
   
   (a) \(b_0\) is a lower bound for \(S\), and
   
   (b) if \(b_1\) is any other lower bound for \(S\), then \(b_1 < b_0\).

We will prove that

- whenever a set \(S\) has a least upper bound, the least upper bound is unique, and
- whenever a set \(S\) has a greatest lower bound, the greatest lower bound is unique.

Thus, it makes sense to talk about THE upper bound or THE lower bound when they exist. Common notation:

- \(\text{lub}(A)\) stands for the least upper bound of \(A\)
- \(\text{glb}(A)\) stands for the greatest lower bound of \(A\)

Does the empty set have a least upper bound or a greatest lower bound?

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\(^8\) the length \(d\) of the diagonal of a unit square
The vocabulary needed to state the Completeness Axiom is now in place. We noted above that the empty set \( \emptyset \) has upper bounds but no least upper bound and has lower bounds but no greatest lower bound. The content of the axiom of completeness is that \( \emptyset \) is the only subset of \( \mathbb{R} \) that exhibits this sort of bad behavior.

**Axiom of Completeness:** Let \( S \) be a nonempty subset of \( \mathbb{R} \).

1. If \( S \) has upper bounds, then \( S \) has a least upper bound.
2. If \( S \) has lower bounds, then \( S \) has a greatest lower bound.

**Example.** I will use the Completeness Axiom to show that there is a positive real number whose square equals 2. Let \( S \) be the set \( \{ x \in \mathbb{R} : x^2 < 2 \} \). \( S \) is bounded above; it is not hard to see\(^9\) that

\[
b \text{ is an upper bound for } S \iff b > 0 \text{ and } b^2 \geq 2.
\]  

(1)

Since many numbers satisfy the condition given in (1), \( S \) has upper bounds, so by the Axiom of Completeness, \( S \) has a least upper bound \( b_0 \).

I will show that \( b_0^2 = 2 \).

1. **First:** suppose, to get a contradiction, that \( b_0^2 < 2 \). Choose any number \( h \) such that \( 0 < h < 1 \) and such that \( h < \frac{2 - b_0^2}{2b_0 + 1} \), and put \( b_1 = b_0 + h \). Then \( b_0 < b_1 \), but

\[
b_1^2 = (b_0 + h)^2 = b_0 + (2b_0 + h)h < (2b_0 + 1)h < b_0^2 + (2 - b_0)^2 = 2,
\]

so that \( b_1 \) is an element of \( S \). But \( b_1 > b_0 \), so that \( b_0 \) is not an upper bound for \( S \) (contradiction).

2. **Next:** suppose, to get a contradiction, that \( b_0^2 > 2 \). In this case, put

\[
b_1 = b_0 - \frac{b_0^2 - 2}{2b_0} = \frac{b_0}{2} + \frac{1}{b_0}.
\]

Then \( 0 < b_1 < b_0 \), but

\[
b_1^2 = \left( b_0 - \frac{b_0^2 - 2}{2b_0} \right)^2 = b_0^2 - \left( b_0^2 - 2 \right) + \left( \frac{b_0^2 - 2}{2b_0} \right)^2 > b_0^2 - (b_0^2 - 2) = 2.
\]  

(2)

Calculation (2) shows that \( b_1^2 > 2 \); since \( 0 < b_1 \), in addition \( 0 < b_1 \), by (1), \( b_1 \) is an upper bound for \( S \). Finally, since \( b_1 < b_0 \), \( b_0 \) is not the least upper bound of \( S \) (contradiction).

Taken together, contradictions 1 and 2 leave \( b_0^2 = 2 \) as the only remaining possibility.

3 **Dedekind Cuts.**

The axiom system for the real numbers works very well, but using an axiom system to define \( \mathbb{R} \) leaves many mathematicians uneasy: after all, the axioms are defining \( \mathbb{R} \) into existence—it feels like wish-fulfillment! Is it really OK to do this?

One potential fix for this problem might be to use a “startup set”—a set that you already have at your disposal—and use it somehow to construct a “final set,” which can then be proved/ to satisfy all of the axioms discussed above. In fact, this program has been carried out twice that I know of; by far the cooler of these constructions is due to Richard Dedekind. I do not propose to show you his construction in detail, merely to indicate in sweeping terms how it works.

Dedekind’s “startup set” is \( \mathbb{Q} \), the set of rational numbers. His “final set” is based upon two insights:

1. **Any** real number—0, \( \sqrt{2} \), \( \pi \), whatever—is completely determined by the set of rational numbers that are less than it.\(^{10}\) (He called the set of all rationals that are less than a given real number a “cut” in \( \mathbb{Q} \).)

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\(^9\) Axioms 6–9 can be used to prove this.

\(^{10}\) Step#5 in the pairing in §4 is based upon this observation.
2. It is possible to describe all cuts in $\mathbb{Q}$ \textit{from within} $\mathbb{Q}$—that is, without ever making reference to irrational numbers. (Thus, for example, he found a way to define the set of rationals that are less than $\sqrt{2}$ without invoking $\sqrt{2}$.)

His “final set” is the set $\mathcal{D}$ of all cuts in $\mathbb{Q}$ (defined, let me reiterate, from within $\mathbb{Q}$). He defines an addition on $\mathcal{D}$, a multiplication on $\mathcal{D}$, and an order on $\mathcal{D}$, and he proves as theorems all of the statements that we are taking as axioms.

4 A Final Note on the Axioms.

Return to footnote 3 on p.1. It turns out that the ten axioms for $\mathbb{R}$ do indeed determine the structure of the real numbers completely, in the sense that if you find two systems $\mathbb{R}$ and $\mathbb{R}'$ that satisfy these axioms, then there is a way of matching up the elements of $\mathbb{R}$ and $\mathbb{R}'$ that allows $\mathbb{R}'$ to be viewed as a relabeling of $\mathbb{R}$. In rough outline, you can find the pairing $\mathbb{R} \leftrightarrow \mathbb{R}'$ in the following steps.

1. Pair $0 \leftrightarrow 0'$ and $1 \leftrightarrow 1'$.

2. (positive integers) For $n \geq 2$, pair $\frac{n}{n} = 1 + \cdots + 1 \leftrightarrow \frac{n'}{n} = 1' + \cdots + 1'$.

3. (negative integers) For $n \geq 1$, pair $-n \leftrightarrow -(n')$.

4. (rational numbers) Pair $\frac{p}{q} \leftrightarrow \frac{p'}{q'}$.

5. (irrationals)\[^{11}\] work for rational $x$ as well, but it isn’t needed. For any irrational $x$, let $\mathcal{C}_x = \left\{ \frac{p}{q} : \frac{p}{q} < x \right\}$; it is not hard to prove that $x = \text{lub}(\mathcal{C}_x)$. Pair $x \leftrightarrow x'$, where $x' = \text{lub}\left( \left\{ \frac{p'}{q'} : \frac{p}{q} \in \mathcal{C}_x \right\} \right)$.

\[^{11}\]This would