A formula for the sign of a permutation

There is a useful formula for the sign of a permutation that is not among the theorems of Chapter 5 (although it probably appears among the exercises someplace). The purpose of this handout is to state and prove this formula.

**Definition.** Let \( \sigma \in S_N \). The **sign** of \( \sigma \), denoted \( (-1)^\sigma \), is defined to be

\[
(-1)^\sigma := \begin{cases} 
1, & \text{if } \sigma \text{ is an even permutation,} \\
-1, & \text{if } \sigma \text{ is an odd permutation.}
\end{cases}
\]

The definition and notation above are standard; for the purposes of this handout, I want to introduce the following nonstandard notation. Let

\[
E(\sigma) := \text{the number of even-length disjoint cycles in } \text{DCF}(\sigma); \quad \text{and}
\]

\[
O(\sigma) := \text{the number of odd-length disjoint cycles in } \text{DCF}(\sigma).
\]

\[
T(\sigma) := \text{the total number of disjoint cycles in the factorization of } \sigma.
\]

Obviously, \( E(\sigma) + O(\sigma) = T(\sigma) \). Note that you include fixed points, which are cycles of length one, when computing \( O(\sigma) \) and in \( T(\sigma) \). Thus, for example, if \( \sigma \in S_{11} \) has disjoint cycle factorization

\[
\sigma = (1, 2)(3, 4, 5)(6, 7, 8, 9)(10)(11),
\]

then \( E(\sigma) = 2, O(\sigma) = 3, \) and \( T(\sigma) = 5 \).

Next, I need to make some preliminary observations; I’ll gather them together into the proposition below.

**Proposition 1.**

[i], for any integers \( n \) and \( k \),

\[
n \equiv k \pmod{2} \implies (-1)^n = (-1)^k;
\]

[ii], for any integer \( n \),

\[
(-1)^n = (-1)^{-n};
\]

[iii],

\[
O(\sigma) \equiv n \pmod{2}.
\]

**Proof.**

[i]: We’re given \( n \equiv k \pmod{2} \), so \( n - k = 2\ell \) for some integer \( \ell \). But then

\[
\frac{(-1)^n}{(-1)^k} = (-1)^{n-k} = (-1)^{2\ell} = ((-1)^2)^\ell = 1^\ell = 1.
\]

[ii]: This is a special case of [i].

[iii]: Note that

\[
n = \sum_{\text{all cycles}} \text{(length of cycle)} = \sum_{\text{even-length cycles}} \text{(length of cycle)} + \sum_{\text{odd-length cycles}} \text{(length of cycle)}. \tag{A} \tag{B}
\]

Now, sum (A), being a sum of even numbers, is even; so \( n \) must be congruent to sum (B) \( \pmod{2} \). But (B) is a sum of odd numbers; so sum (B) is odd if and only if the number of terms in the sum—which is \( O(\sigma) \)—is odd. Thus,

\[
n \text{ is } \begin{cases} \text{even} \implies \text{sum(B) is even} \implies O(\sigma) \text{ is even} \end{cases} \]

\[
n \text{ is } \begin{cases} \text{odd} \implies \text{sum(B) is odd} \iff O(\sigma) \text{ is odd} \end{cases} \]

\[\blacksquare\]
As a final preliminary, I want to break off the first step in the proof of the theorem as Proposition 2. That will allow the proof of the theorem to be an uninterrupted flowing computation.

**Proposition 2.** $(-1)^\sigma = (-1)^{E(\sigma)}$.

**Proof.** Since even-length cycles are odd permutations and vice-versa, $\sigma$ is an \{even\} permutation if and only if the number of even-length cycles in $DCF(\sigma)$ is an \{even\} number—that is, if and only if $E(\sigma)$ is an \{even\} number. ■

I am now ready to state and prove the theorem.

**Theorem.** Let $\sigma \in S_n$. With the notation as above,

$$(-1)^\sigma = (1)^{n-T(\sigma)}.$$  

**Proof.** The proof is the following computation:

\[ (-1)^\sigma \]

by Proposition 2 $\rightarrow= (-1)^{E(\sigma)}$  

since $E(\sigma) + O(\sigma) = T(\sigma)$ $\rightarrow= (-1)^{T(\sigma) - O(\sigma)}$  

by Proposition 1, parts [i] and [iii] $\rightarrow= (-1)^{T(\sigma) - n}$  

by Proposition 1, part [ii] $\rightarrow= (-1)^{n - T(\sigma)}$. ■