1. Use contour integration to show that the integral of \( f(z) = \bar{z} \) around any circle in the complex plane (taken counterclockwise) is equal to \( 2i \) times the area of that circle.

**Solution.** The circle \(|z - z_0| = R\) can be parametrized \( z(t) = z_0 + Re^{it} \), for \( 0 \leq t \leq 2\pi \). Then \( \int_C \bar{z} \, dz = \int_0^{2\pi} \overline{z_0 + Re^{it}} \cdot iRe^{it} \, dt = \int_0^{2\pi} \overline{z_0} iRe^{it} + iR^2 \, dt = \overline{z_0} Re^{it} + tiR^2 \big|_0^{2\pi} = 2\pi iR^2 \)

2. Find the integral of \( f(z) = \text{P.V.} z^{1/3} \sin(-e^{z^2}) \) counterclockwise around the circle of radius 1 centered at 2 + 2i.

**Solution.** Since the contour does not intersect the negative real line or the origin, \( \text{P.V.} \, z^{1/3} \) is analytic inside and on the contour. All the other functions used to compose \( f(z) \) are entire, so \( f(z) \) is analytic inside and on the contour. Therefore, the contour integral is 0.

3. Let \( f \) be a continuous (but not necessarily analytic) function such that \(|f(z)| > 3\) on the unit circle. Explain how we know that \(|\int_C 1/f(z) \, dz| < 2\pi/3\), where \( C \) is the unit circle, taken in the positive sense.

**Solution.** Since \( f(z) \neq 0 \) on the unit circle, \( 1/f(z) \) is defined and continuous on the circle, and \( 0 < |1/f(z)| < 1/3 \). Using UBM, \( \int_C |1/f(z)| \, dz < (1/3)L \), where \( L \) is the length of \( C \), which is \( 2\pi \). Using MIIM, \( |\int_C 1/f(z) \, dz| \leq \int_C |1/f(z)| \, dz < 2\pi/3 \).

4. Prove that \( f(z) = 1/z \) does not have an antiderivative on the domain \( \mathbb{C} - \{0\} \) (hint: evaluate its integral around the unit circle, and use the antiderivative theorem).

**Solution.** By the CIF, \( \int_C 1/z \, dz = 2\pi i \), so at least one integral around a closed contour in the domain is nonzero. By ANTI, \( 1/z \) does not have an antiderivative throughout \( \mathbb{C} - \{0\} \).

5. Prove that \( f(z) = 1/z \) does have an antiderivative on the upper half plane \( \text{Im} \, z > 0 \).

**Solution.** The upper half plane is simply connected, and \( f(z) \) is analytic there (it’s the quotient of nonzero analytic functions). By CGT, all the integrals around closed contours are zero, and hence by ANTI \( f(z) \) has an antiderivative.

6. Find the Laurent series representations of

\[ f(z) = \frac{1}{z(1-z)}, \quad z \neq 0, z \neq 1 \]

(a) centered at 0,

(b) centered at 1, and

(c) for the domain \(|z| > 1\).

Indicate the largest annulus on which each series converges. In addition, use your Laurent series to find \( \text{Res}_{z=0} f(z) \) and \( \text{Res}_{z=1} f(z) \). What does this tell you about the integral of \( f(z) \) around the circle \(|z| = 2\), taken in the positive sense?
Solution.

(a) We want to expand in powers of $z$, so use

$$f(z) = \frac{1}{z} \cdot (-1 - z - z^2 - z^3 - \cdots) = -z^{-1} - 1 - z - z^2 - \cdots.$$  

The radius of convergence is the distance to the nearest singular point, so it converges for $0 < |z| < 1$.

(b) We want to expand in powers of $z - 1$, so use

$$\frac{1}{z-1} \cdot \frac{1}{1 + (z-1)} = \frac{1}{z-1} (1 - (z-1) + (z-1)^2 - \cdots) = (z-1)^{-1} - 1 + (z-1) - (z-1)^2 + \cdots$$

The radius of convergence is the distance to the nearest singular point, so it converges for $0 < |z - 1| < 1$.

(c) We want to expand in powers of $1/z$, so use

$$\frac{1}{z} \cdot \frac{-1/z}{1-1/z} = \frac{1}{z} (-\frac{1}{z} - \frac{1}{z^2} - \cdots) = -z^{-2} - z^{-3} - z^{-4} - \cdots,$$

which converges when $0 < |1/z| < 1$.

Now, $\text{Res}_{z=0} f(z) = -1$ is the coefficient of $z^{-1}$ in the expansion around 0, and $\text{Res}_{z=1} f(z) = 1$ is the coefficient of $(z-1)^{-1}$ in the expansion around 1. By RES, the integral of $f(z)$ around the circle of radius 2 is $2\pi i$ times the sum of the residues inside the circle, so the integral is 0.

7. Find the Laurent series representation of

$$f(z) = \frac{\sin z}{z}, \quad z \neq 0$$

centered at 0, and indicate the largest annulus on which this series converges.\(^1\) How can we define $f(0)$ so that $f(z)$ is entire (hint: look at the Laurent series)? In addition, use your Laurent series to find $\text{Res}_{z=0} f(z)$. What does this tell you about the integral of $f(z)$ around the circle $|z| = 2$, taken in the positive sense?

Solution. We have $(\sin z)/z = 1 - z^2/3! + z^4/5! - \cdots$ which converges on the entire complex plane. This suggests that we can make $f(z)$ continuous by defining $f(0) \equiv 1$. The residue $\text{Res}_{z=0} f(z)$ is 0 because there is no $z^{-1}$ term in the Laurent series, so the integral around the circle is 0.\(^2\)

---

\(^1\)You can use the fact that $\sin z = z - z^3/3! + z^5/5! - \cdots$.

\(^2\)This discussion shows that we can extend $f(z)$ to an entire function by defining $f(0) \equiv 1$. The singularity at $z = 0$ is called a “removable” singularity.