Introduction to Linear Systems
Differential Equations / Dr. Rachel Hall

Overview The purpose of this handout is to provide you with the background in linear algebra necessary for understanding planar systems of linear differential equations.

Multiplying matrices. The product of a $2 \times 2$ matrix ($2 \times 2$ array of numbers) with a $1 \times 2$ vector is defined by

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.
\]

Then

1. \[
\begin{pmatrix} 2 & 5 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} =
\]

2. \[
\begin{pmatrix} 2 & 5 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} =
\]

3. \[
\begin{pmatrix} 2 & 5 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3e^{-t} - e^t \\ 2e^{-t} + e^t \end{pmatrix} =
\]

(simplify your answer)

4. \[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} =
\]

The matrix \[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
is called the $2 \times 2$ identity matrix. You have shown that if $I$ is the identity matrix and $V$ is any vector of the form $\begin{pmatrix} a \\ b \end{pmatrix}$ then $IV = V$.

Linear systems. Recall that the derivative of a $1 \times 2$ vector $Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, written $dY/dt$, is defined by

\[
\frac{dY}{dt} = \frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix}.
\]

A planar system of linear differential equations, or linear system for short, is defined to be any system that can be written in the form $dY/dt = AY$, where $A$ is a $2 \times 2$ matrix and $Y$ is the vector $(x, y)$ (also written $\begin{pmatrix} x \\ y \end{pmatrix}$).
1. Write the system

\[
\begin{align*}
\frac{dx}{dt} &= 4x + 3y \\
\frac{dy}{dt} &= -x
\end{align*}
\]

in the form \( d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y} \), where \( \mathbf{A} \) is a matrix and \( \mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix} \).

2. Use matrices to show that \( \mathbf{Y}(t) = (6e^{3t}, -2e^{3t}) \) is a solution to the system of differential equations in question 1.

3. Is it possible to write the system

\[
\begin{align*}
\frac{dx}{dt} &= 3x^2 + 5xy \\
\frac{dy}{dt} &= -xy + 7y^2
\end{align*}
\]

in the form \( d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y} \)?

Determinants The determinant of a \( 2 \times 2 \) matrix \( \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), written \( \det \mathbf{A} \), is defined by

\[
\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.
\]

Then

1. If \( \mathbf{A} = \begin{pmatrix} 2 & -6 \\ -1 & 1 \end{pmatrix} \), then \( \det \mathbf{A} = \)

2. If \( \mathbf{A} = \begin{pmatrix} 2 - z & -6 \\ -1 & 1 - z \end{pmatrix} \), then (in simplest form) \( \det \mathbf{A} = \)
The zero vector, \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) is also written \( \mathbf{0} \). We showed in class that the system \( A \mathbf{Y} = \mathbf{0} \) has exactly one solution (\( \mathbf{Y}(t) = \mathbf{0} \) for all \( t \)) if \( \det A \neq 0 \); it has infinitely many solutions if \( \det A = 0 \). This fact proves the theorem:

**Theorem 1** The linear system \( d\mathbf{Y}/dt = A \mathbf{Y} \) has exactly one equilibrium solution, namely, \( \mathbf{Y}(t) = \mathbf{0} \), if and only if \( \det A \neq 0 \).

Discuss how the number of solutions of the system \( A \mathbf{Y} = \mathbf{0} \), where \( A \) is the matrix in problem 2 above, depends on the value of \( z \).

---

**The Linearity Principle**

Compute

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} =
\]

Now compute

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x + z \\ y + w \end{pmatrix} =
\]

and show that your answers are equal.

You have just shown that \( A(\mathbf{Y}_1 + \mathbf{Y}_2) = A\mathbf{Y}_1 + A\mathbf{Y}_2 \). Now if \( d\mathbf{Y}_1/dt = A\mathbf{Y}_1 \) and \( d\mathbf{Y}_2/dt = A\mathbf{Y}_2 \), then

\[
\frac{d}{dt}(\mathbf{Y}_1 + \mathbf{Y}_2) = \frac{d\mathbf{Y}_1}{dt} + \frac{d\mathbf{Y}_2}{dt} = A\mathbf{Y}_1 + A\mathbf{Y}_2 = A(\mathbf{Y}_1 + \mathbf{Y}_2)
\]

using what you have shown. This observation leads us to the linearity principle:
The Linearity Principle

1. if $Y_1(t)$ and $Y_2(t)$ are both solutions to the linear system $dY/dt = AY$, then $Y_1(t) + Y_2(t)$
is also a solution, because $(d/dt)(Y_1 + Y_2) = A(Y_1 + Y_2)$.

2. if $Y(t)$ is a solution to the linear system $dY/dt = AY$, then $kY(t)$ is also a solution for any
constant $k$. This can be shown by similar methods.

(A trick question) What does the linearity principle tell you about solutions to the system

\[
\begin{align*}
\frac{dx}{dt} &= 3x^2 + 5xy \\
\frac{dy}{dt} &= -xy + 7y^2
\end{align*}
\]

The general solution.\(^1\) If $Y_1(t)$ and $Y_2(t)$ are two solutions of the two-variable linear system
$dY/dt = AY$ that are linearly independent (in systems of two variables, this just means that
they’re not multiples of each other), then the general solution of the system is

\[Y(t) = k_1Y_1(t) + k_2Y_2(t).\]

1. Suppose $Y_1(t) = \begin{pmatrix} e^t \\ e^{2t} \end{pmatrix}$ and $Y_2(t) = \begin{pmatrix} -e^{2t} \\ 3e^{2t} \end{pmatrix}$ are both solutions to the linear system
d$Y/dt = AY$. Write down the general solution to the system, and use it to solve the initial
value problem $Y(0) = (0,5)$.

2. Suppose $Y_1(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$ and $Y_2(t) = \begin{pmatrix} -2e^{3t} \\ -4e^{3t} \end{pmatrix}$ are both solutions to the linear system
d$Y/dt = AY$. What is the general solution? Can you solve the initial value problem
$Y(0) = (0,5)$?

\(^1\)How would the general solution change for a system of three variables? A system of $n$ variables?
3. Why is the condition “not multiples of each other” (i.e. linearly independent) important?

**Straight-line solutions** We now need to find two linearly independent solutions $Y_1(t)$ and $Y_2(t)$. Then we can use the linearity principle to write down the general solution.

Here is a picture of the system 

$$ \frac{dY}{dt} = \begin{pmatrix} -4 & -2 \\ -1 & -3 \end{pmatrix} Y. $$

There are two families of solutions that lie on straight lines. We just need to find two linearly independent solutions, so let’s find the equations for these straight-line solutions!

Mark two points on the right-hand graph: $(0, 1)$ and $(-1, 1)$. For each point $(x, y)$, (1) draw the vector $(x, y)$, and (2) draw the vector $dY/dt|_{(x,y)}$ at $(x, y)$.

Observe that a point $(x, y)$ lies on a straight-line solution to $dY/dt = AY$ if and only if the vector $dY/dt = (ax + by, cx + dy)$ is a multiple of the vector $(x, y)$; that is, for some value $\lambda$,

$$ \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}, $$

which implies

$$ \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. $$

If $V = (x, y)$, then equation (1) becomes $AV = \lambda V$, and equation (2) becomes

$$ (A - \lambda I)V = 0. $$
The number of solutions to the system \((A - \lambda I)V = 0\) is determined by the determinant of the matrix \((A - \lambda I)\). If the matrix has nonzero determinant, then there is only one solution, \(V = 0\). If the determinant of \((A - \lambda I)\) is zero, then there are multiple solutions.

Remember that we are trying to find the straight-line solutions \(Y(t)\) to the original system of DEs. If the determinant of \((A - \lambda I)\) is nonzero, the only solution to the problem is \(V = 0\), which doesn’t help! So we want the determinant of \((A - \lambda I)\) to be zero.

Let’s see what this will buy us. Suppose you have found a value of \(\lambda\) such that \(AV = \lambda V\) has nonzero solutions, and suppose you’ve chosen an actual nonzero vector \(V\) that satisfies the equation. Show that \(Y(t) = e^{\lambda t}V\) (a lucky guess!) is a solution to \(dY/dt = AY\).

\[
dY/dt = (d/dt)(e^{\lambda t}V) = \\
AY = A(e^{\lambda t}V) = 
\]

**Method for finding straight-line solutions (if they exist).** Suppose we want to find straight-line solutions to the system \(dY/dt = AY\).

1. Solve the equation \(\det(A - \lambda I) = 0\) for \(\lambda\). The possible values of \(\lambda\) are called *eigenvalues*.

2. For each value of \(\lambda\) you obtain, solve the system \((A - \lambda I)V = 0\) for \(V\). These vectors \(V\) give the direction of the straight-line solutions to the original equation and are called *eigenvectors*.

3. Choose any nonzero \(V\) that is an eigenvector for \(\lambda\). Then \(Y(t) = e^{\lambda t}V\) is a straight-line solution to the original system of DEs.

**Example.** Find the straight-line solutions to

\[
\frac{dY}{dt} = \begin{pmatrix} -4 & -2 \\ -1 & -3 \end{pmatrix} Y.
\]

1. Find the eigenvalues.

\[
\det(A - \lambda I) = \det \begin{pmatrix} -4 - \lambda & -2 \\ -1 & -3 - \lambda \end{pmatrix} = \\
Show that the eigenvalues are \(\lambda_1 = -5\) and \(\lambda_2 = -2\).
2. Find an eigenvector for each eigenvalue. For $\lambda_1 = -5$, this is done by solving
\[
(A - (-5)I)V = 0
\]
for $V$. So,
\[
\begin{pmatrix}
-4 & -2 \\
-1 & -3
\end{pmatrix} - \begin{pmatrix}
-5 & 0 \\
0 & -5
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]
\[
\begin{pmatrix}
1 & -2 \\
-1 & 2
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]
\[
\begin{pmatrix}
x - 2y \\
x + 2y
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]
Any vector $(x, y)$ that satisfies $x - 2y = 0$ is a solution. We only need one, so let’s pick (2, 1). The vector $V_1 = (2, 1)$ is an eigenvector for the eigenvalue $\lambda_1 = -5$. Now, find an eigenvector $V_2$ for the eigenvalue $\lambda_2 = -2$.

3. One straight-line solution is $Y_1(t) = e^{\lambda_1 t}V_1 = e^{-5t}(2, 1) = (2e^{-5t}, e^{-5t})$. Find another solution that is linearly independent from $Y_1$. 


The general solution. Finally, let’s put it all together. If we are able to find two straight-line solutions $Y_1(t)$ and $Y_2(t)$ to $dY/dt = AY$ that are not multiples of each other, then the general solution is\(^2\)

$$Y(t) = k_1Y_1(t) + k_2Y_2(t).$$

1. Write down the general solution to the problem on the previous three pages.

2. Check that it is a solution to $dY/dt = AY$.

3. Solve the initial value problem $Y(0) = (0, 5)$.

\(^2\)Prove that this is the general solution (that is, first prove that it is a solution, then prove that it allows you to solve every initial-value problem). What does the general solution of a $n \times n$ linear system look like? How many eigenvalues will you need to find? How many linearly independent eigenvectors?