Final Exam
Discrete Structures
St. Joseph's University • December 18th, 2000

Directions. There are 7 pages. Problems are worth 10 points except where indicated; the total points possible are 150 points. This exam contains three parts: the first part contains short-answer questions for which work need not be shown, the second part contains longer questions for which you must show work, and the third part contains proofs. Good luck!!

Part I: Short-answer questions. (70 points) You need not show work.

1. Let \( X = \{a, b, c\} \) and \( Y = \{1, 2, 3, 4\} \).

(a) Add arrows to the diagram below to define a function \( f : X \rightarrow Y \) that is one-to-one, but not onto.

(b) Add arrows to the diagram below to define a function \( g : X \rightarrow Y \) that is neither one-to-one nor onto.

2. Give an example of finite sets \( A, B, \) and \( C \) and functions \( f : A \rightarrow B \) and \( g : B \rightarrow C \) such that their composition \( g \circ f \) is a one-to-one correspondence but neither \( f \) nor \( g \) are one-to-one correspondences. Use arrows to define \( f \) and \( g \).
3. Let $A = \{1, 2, 3, 4, 5, 6\}$ and suppose $R$ is the equivalence relation on $A$ whose equivalence classes are:

\[
\begin{align*}
\{1, 2, 3, 4\} &= [1] \\
\{5\} &= [5] \\
\{6\} &= [6]
\end{align*}
\]

Draw the directed graph of $R$ by adding arrows to the diagram below:

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1 •     2 •

6 •     3 •

5 •     4 •
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4. (15 points) Write negations for the following statements:

(a) Montgomery Burns is rich and happy. (Use the word “or” in your negation.)

(b) Every dog has his day.

(c) If $r$ is an irrational number, then $r^3 - r^2$ is an irrational number.

5. A standard set of 52 playing cards contains cards of four suits ($\bullet$, $\heartsuit$, $\diamondsuit$, and $\clubsuit$) and 13 denominations (a number or a face card). How many cards must be drawn in order to be sure of drawing

(a) two cards of the same suit?
(b) three cards of the same suit?
(c) four cards of the same denomination?
6. (15 points) The following statements are **false**. Find a counterexample for each statement.

   (a) The product of a rational number and an irrational number is irrational.

   (b) If $A$, $B$, and $C$ are sets, then $A - (B - C) = (A - B) - C$.

   (c) For all real numbers $x$ and $y$, $|x| + |y| = |x + y|$.

**Part II: Longer questions.** (30 points) The following questions will be graded on a partial credit basis. Show all your work clearly and in detail. Answers which are unsupported by work will **not** receive credit.

7. Let $k$ be a positive integer. Show that if $\sum_{i=1}^{k}(2i - 1) = k^2$, then $\sum_{i=1}^{k+1}(2i - 1) = (k + 1)^2$. 
8. Define the relation $R$ from $\mathbb{R}$ to $\mathbb{Z}$ by

$$\text{for all } (x, y) \in \mathbb{R} \times \mathbb{Z}, \quad (x, y) \in R \iff \lfloor x \rfloor = y.$$  

Find $R^{-1}(4)$, and use the definition of $R^{-1}$ to justify your answer.

9. Define functions $F$ and $G$ from $\mathbb{R} \times \mathbb{R}$ to itself by the formulas:

$$\text{for all } (a, b) \in \mathbb{R} \times \mathbb{R},$$

$$F(a, b) = (a + b, b) \quad \text{and} \quad G(a, b) = (a + 2b, b).$$

(a) Find $F(3, 1)$, $G(1, 1)$, $(F \circ G)(1, 1)$, and $(G \circ F)(5, 0)$.

(b) Show that $F \circ G = G \circ F$. 


Part III: Proofs. (48 points) Do only four of the following five proofs. They will be graded on a partial credit basis. Each proof is worth 12 points. Show all your work clearly and in detail. You must leave one question blank.

10. Prove that the function $g : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by the formula $g(n) = 2n - 4$, for all integers $n$, is one-to-one.

11. The congruence modulo 5 relation, $R$, is defined from $\mathbb{Z}$ to $\mathbb{Z}$ as follows:

   for all integers $m$ and $n$, $\ m \ R \ n \ \iff \ 5 \mid (m - n)$.

   Prove that $R$ is transitive.
12. Suppose $R$ is the relation from $\mathbb{R}$ to $\mathbb{R}$ defined by

   for all real numbers $x$ and $y$, \[ x \, R \, y \iff x + xy = 2 - y. \]

   Prove that $R$ is symmetric.

13. Let $R$ be an equivalence relation on a set $A$ and let $a$ and $b$ be elements of $A$. Prove that if $a$ is not related to $b$ by $R$ then $[a] \cap [b] = \emptyset$. 
14. Let $A = \{1, 2, 3, \ldots, 10\}$ and let $\mathcal{P}(A)$ be the power set of $A$. Define the function $f : \mathcal{P}(A) \to \{0, 1, 2, \ldots, 10\}$ by

$$f(X) = \text{the number of elements in } X.$$  

Prove that $f$ is onto.