Asymmetric Rhythms and Tiling Canons

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Abstract

A musical rhythm pattern is a sequence of note onsets. We consider repeating rhythm patterns, called rhythm cycles. Many typical rhythm cycles from Africa are asymmetric, meaning that they cannot be broken into two parts of equal duration. More precisely: if a rhythm cycle has a period of $2n$ beats, it is asymmetric if positions $x$ and $x+n$ do not both contain a note onset. We ask the questions (1) How many asymmetric rhythm cycles of period $2n$ are there? (2) Of these, how many have exactly $r$ notes? We use Burnside’s lemma to count these rhythms. Our methods also answer analogous questions involving division of rhythm cycles of length $\ell n$ into $\ell$ equal parts. Asymmetric rhythms may be used to construct rhythmic tiling canons—that is, canons in which there is exactly one note onset per beat. Our results count rhythmic tiling canons where the voice entries are equally spaced. Audio recordings of all examples discussed are available at http://www.sju.edu/~rhall/Rhythms.

1 Introduction

If you walk into your neighborhood record store, you will be confronted with an array of different musical genres. What makes a musical style distinctive? Certainly, instrumentation is important: one does not expect to hear a trumpet playing bluegrass or a banjo in a mariachi band. However, drums and guitar are almost ubiquitous in popular music around the world; instrumentation is clearly not the whole story. Our own personal likes and dislikes are strongly influenced by the rhythms, melodic structures, harmonies, and

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lyrics in the songs we hear. This article focuses on what may be the most important of these aspects: rhythm. We examine the mathematics of some rhythmic structures common in popular and folk music.

Anyone who listens to rock music is familiar with the repeated drum beat—one, two, three, four—based on a 4/4 measure. Fifteen minutes listening to a Top 40 radio station is evidence enough that most rock music has this basic beat. But if we tune the radio to different frequencies, we may hear popular music (jazz, Latin, African) with different characteristic rhythms. Although much of this music is also based on the 4/4 measure, some instruments play repeated patterns that are not synchronized with the 4/4 beat, creating syncopation—an exciting tension between different components of the rhythm. This article is concerned with classifying and counting rhythms that are maximally syncopated in the sense that, even when shifted, they cannot be synchronized with the division of a measure into two parts. In addition, we discuss rhythms that cannot be aligned with other even divisions of the measure. Our results have a surprising application to rhythmic canons.

2 Rhythm patterns, rhythm cycles, and asymmetry

Let us introduce some of the musical and mathematical definitions central to this article.

2.1 Rhythm patterns and cycles

A rhythm pattern is a sequence of note onsets. We assume there is some basic, invariant unit pulse or beat that cannot be divided; that is, every note onset occurs at the beginning of some pulse. We identify two rhythm patterns if they have the same sequence of onsets. For example,

\[ \text{\mid \mid \mid} \quad \text{and} \quad \text{\textsuperscript{7\textsuperscript{7}\textsuperscript{7}\textsuperscript{7}\textsuperscript{7}}} \text{are equivalent.} \]

Here, we consider only periodic rhythm patterns. In this case, it is natural to deem two rhythms equivalent if one is a shift of the other. For example,

\[ |: \text{\textsuperscript{7\textsuperscript{7}\textsuperscript{7}\textsuperscript{7}\textsuperscript{7}}} :| \quad \text{is equivalent to} \quad |: \text{\textsuperscript{7\textsuperscript{7}\textsuperscript{7}\textsuperscript{7}\textsuperscript{7}}} :| \]

In this article, repeat signs (“|” and “:”) indicate periodic rhythms. We call the equivalence classes rhythm cycles. We sometimes call one period of the cycle a measure.
Here are five different notations for the same rhythm cycle.

<table>
<thead>
<tr>
<th>notation</th>
<th>notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>standard</td>
<td></td>
</tr>
<tr>
<td>drum tablature</td>
<td>x..x..x..x..</td>
</tr>
<tr>
<td>binary</td>
<td>10010010</td>
</tr>
</tbody>
</table>

The first line shows the standard Western musical notation. Since only note onsets, not durations, matter, we can represent the same pattern using x’s for note onsets and .’s for rests—a notation often used by drummers. Binary notation replaces the x’s by 1’s and the .’s by 0’s. An especially suggestive notation is the representation of rhythm cycles as necklaces of black and white beads, with black beads corresponding to note onsets and white ones to rests. In this case, the cyclic shift becomes a rotation. There is extensive literature on such binary necklaces, to which our results contribute.

2.2 Asymmetry and syncopation

Many rhythm cycles from Africa, Latin America, and Eastern Europe are asymmetric—that is, they cannot be broken into two parts of equal duration, where each part starts with a note onset (see Arom [4, pp. 243–245]). The rhythm cycle |: x..x..x..x:| is asymmetric; however, the cycle |: x.x.x..x.| is not, as it is equivalent to |: x.x..x|x.:|, which has a note onset at both the beginning and the midpoint of the measure.

Asymmetry is closely related to syncopation, defined by the New Harvard Dictionary of Music [13, p. 827] as “a momentary contradiction of the prevailing meter or pulse.” According to this source, syncopation can take two forms:

1. “a temporary transformation of the fundamental character of the meter, e.g., from duple to triple,” and

2. “the contradiction of the regular succession of strong and weak beats within a measure.”

For the most part, we are concerned with (2). In duple meter, strong beats occur at the beginning and midpoint of a measure. Asymmetric rhythm cycles are, in a sense, maximally syncopated in duple meter: although they live in a world in which measures are naturally divided in half, they cannot be delayed so that note onsets coincide with both of the strong beats within the measure. Asymmetric rhythms are always a little out of sync with our expectations.
2.3 Rhythms as functions

We now translate into mathematical terms. A rhythm pattern can be represented as a function \( f : \mathbb{Z} \to \{0,1\} \), where \( f(x) = 1 \) if there is a note onset on pulse \( x \) and \( f(x) = 0 \) otherwise. The function \( f \) represents a periodic rhythm of period \( p \) if \( f(x) = f(x+p) \) for all \( x \in \mathbb{Z} \); in this case, \( f \) can be identified with a function with domain \( \mathbb{Z}/p\mathbb{Z} \) or \( \mathbb{Z}_p \). A rhythm cycle is defined to be an equivalence class of functions on \( \mathbb{Z}_p \) modulo rotation. That is, \( f_1 \) is equivalent to \( f_2 \) if for some \( k \) and for all \( x \), \( f_1(x) = f_2(x-k) \).

Finally, we want to consider not all periodic rhythm patterns but only those that are asymmetric—a notion that makes sense only if the period is even. A rhythm pattern of period \( p = 2n \) is asymmetric modulo \( 2n \) if its corresponding function \( f : \mathbb{Z}_{2n} \to \{0,1\} \) does not have two onsets that are separated by half a measure (that is, \( f(x) = 1 \) implies that \( f(x+n) = 0 \) for all \( x \in \mathbb{Z}_{2n} \)).

3 Counting asymmetric rhythm cycles

We begin by gathering the asymmetric patterns of period \( 2n \) into a set:

\[
S^n_2 := \{ f : \mathbb{Z}_{2n} \to \{0,1\} \mid f(x) = 1 \Rightarrow f(x+n) = 0 \}.
\]

In total, \( S^n_2 \) contains \( 3^n \) functions. Indeed, if we partition the elements of \( \mathbb{Z}_{2n} \) into \( n \) pairs \( \{\{0,n\}, \{1,n+1\}, \ldots \{n-1,2n-1\}\} \), then constructing a function \( f \in S^n_2 \) corresponds to choosing, for each pair, one of the following three possibilities:

**Choice 1.** \( f = 0 \) for both members of the pair.
**Choice 2.** \( f = 1 \) for the first element and \( f = 0 \) for the second element. \((*)\)
**Choice 3.** \( f = 0 \) for the first element and \( f = 1 \) for the second element.

We count the total number of asymmetric rhythm cycles by starting with the set \( S^n_2 \) and counting the number of equivalence classes modulo cyclic shifts (rotations). Similarly, for each \( 0 \leq r \leq n \) (\( n \) being the maximum possible number of onsets for an asymmetric rhythm pattern) we count the number of asymmetric rhythm cycles of period \( 2n \) with \( r \) note onsets by starting with the subset

\[
S^n_2(r) := \{ f \in S^n_2 \mid |f^{-1}(\{1\})| = r \},
\]

and counting equivalence classes modulo cyclic shifts.

In both cases, the equivalence classes are orbits induced by a group action (recall: if group \( G \) acts on set \( S \) and \( s \in S \), the orbit of \( s \) is the set \( \{g \cdot s \mid g \in G\} \)). For rhythm cycles, the
group is $\mathbb{Z}_{2n}$, and element $m \in \mathbb{Z}_{2n}$ acts on a pattern of period $2n$ by shifting it through $m$ positions. On the level of functions: for $f \in S^n_2$ (respectively $f \in S^n_2(r)$), the function $m \cdot f$ is given by $(m \cdot f)(x) = f(x - m)$. Because the equivalence classes are orbits, we can apply Burnside’s lemma. The statement of this lemma is as follows; for a proof, see [5, p. 563].

**Burnside’s Lemma** Let a finite group $G$ act on a finite set $S$; for each $\beta \in G$, define $\text{fix}(\beta)$ to be the number of elements $s \in S$ such that $\beta \cdot s = s$. Then the number of orbits that $G$ induces on $S$ is given by

$$\frac{1}{|G|} \sum_{\beta \in G} \text{fix}(\beta).$$

### 3.1 The total number of asymmetric rhythm cycles

Theorems 1 and 2 originally appeared in our paper [11].

**Theorem 1** The number of asymmetric rhythm cycles of period $2n$ is given by

$$\frac{1}{2n} \left[ \sum_{d|n} \phi(2d) + \sum_{d|n \text{ odd}} \phi(d)3^{n/d} \right],$$

(1)

where $\phi(d)$ is the number of integers $1 \leq x \leq d$ such that $x$ is relatively prime to $d$.

To prove this theorem, we let the group $\mathbb{Z}_{2n}$ act on the set $S^n_2$, and we count the orbits (i.e. cycles) using Burnside’s lemma. This theorem is the restriction of Theorem 5 to the case $\ell = 2$; a full statement and proof of Theorem 5 is provided below.

$\square$

### 3.2 The number of $r$–note asymmetric rhythm cycles

The approach here is analogous to that in the previous section; we let $\mathbb{Z}_{2n}$ act on $S^n_2(r)$, and we count the orbits. This theorem is the case $\ell = 2$ of Theorem 7, which is stated and proved below.
Theorem 2 For any $1 \leq r \leq n$, the number of asymmetric rhythm cycles with $r$ note onsets is given by

$$
\frac{1}{2n} \sum_{d | \gcd(n,r) \atop d \text{ odd}} \phi(d) \left( \frac{n}{d} \right) \left( \frac{r}{d} \right)^2 2^{r/d}.
$$

(2)

3.3 The number of self-complementary asymmetric rhythm cycles

In general, the complement $f^c$ of a rhythm pattern $f$ is formed by exchanging note onsets and rests; that is, $f^c = 1 - f$. A pattern of period $p$ is self-complementary if $f$ and $f^c$ are in the same rhythm cycle. For example, x.x.x.x.—alternately 11010010—is self-complementary. This occurs only when the shortest period of $f$ is even, so that every period of $f$ is even; and a moment’s reflection shows that an asymmetric rhythm pattern $f$ of period $2n$ is self-complementary if and only if $|f^{-1}(\{1\})| = n$. Thus, one finds the number of self-complementary asymmetric cycles by putting $r := n$ in Equation 2:

Corollary 3 The number of self-complementary asymmetric rhythm cycles is given by

$$
\frac{1}{2n} \sum_{d | n \atop d \text{ odd}} \phi(d) 2^{n/d}.
$$

(3)

Corollary 3 is not a new result. It appears in [10, p. 172] (reported in [14]) as the solution to a problem easily seen to be equivalent to this one. It is sequence A000016 in [15].

4 Generalization to $\ell$-asymmetry

Our characterization of asymmetry as “maximal syncopation” was based on division of the measure into two parts (duple meter). Returning to Section 2.2, we see that syncopation can occur in any meter—suggesting that we should expand our definition of asymmetry. Suppose we divide a measure of $M$ beats into $\ell$ equal parts and place a strong beat at the beginning of each part, creating “$\ell$-tuple meter.” One can find rhythm cycles that are maximally syncopated in $\ell$-tuple meter: they cannot be broken into $\ell$ parts of equal duration, where more than one part starts with a note onset. We say that a periodic
rhythm of period \( \ell n \) is \( \ell \)-asymmetric if when position \( x \) contains a note onset, then all other positions \( y \), where \( y \equiv x \pmod{n} \), do not contain note onsets. For example, the 12-periodic rhythm \( \cdots x \cdots x \cdots x \cdots x : \) is 3-asymmetric \((n = 4)\). No matter how it is shifted, one cannot divide the measure into three equal parts such that more than one part starts with a note onset. In other words, all of these shifts are syncopated in triple meter. However, this rhythm clearly is not 2-asymmetric, and hence neither 4-asymmetric nor 6-asymmetric.

Note that our previous definition of asymmetry corresponds to \(-\)asymmetry when \( \ell = 2 \).

Let \( n, \ell \geq 2 \), and set \( M := \ell n \). For any divisor \( y \) of \( M \) and any \( x \in \mathbb{Z}_M \), let
\[
[x]_y := \{ z \in \mathbb{Z}_M : z \equiv x \pmod{y} \}.
\]

We are interested in functions \( f : \mathbb{Z}_M \rightarrow \{0, 1\} \) that never assign a 1 to two elements that are congruent modulo \( n \). Put
\[
S^n_\ell := \{ f : \mathbb{Z}_M \rightarrow \{0, 1\} \mid (x \equiv y \pmod{n} \text{ and } x \neq y) \Rightarrow f(x) + f(y) \leq 1 \}.
\]

Clearly \(|S^n_\ell| = (\ell + 1)^n\), for such a function is constructed by choosing, from each equivalence class \([x]_n\), either zero or one element to be mapped to 1.

We also examine, for \( 0 \leq r \leq n \), the subset of rhythms with \( r \) note onsets:
\[
S^n_\ell(r) := \{ f \in S^n_\ell : |f^{-1}(\{1\})| = r \}.
\]

Clearly, \(|S^n_\ell(r)| = \binom{n}{r} \ell^r\).

In each case, we let \( \mathbb{Z}_M \) act on the set of functions and count the orbits. We let \( R^n_\ell \) (respectively \( R^n_\ell(r) \)) denote the set of orbits of \( S^n_\ell \) (respectively \( S^n_\ell(r) \)) modulo cyclic shifts. In other words,
\[
R^n_\ell := S^n_\ell / \mathbb{Z}_M \quad \text{and} \quad R^n_\ell(r) := S^n_\ell(r) / \mathbb{Z}_M.
\]

We determine \(|R^n_\ell| \) and \(|R^n_\ell(r)| \) in Theorems 5 and 7. These results generalize Theorem 1 (respectively, Theorem 2), which gives \(|R^n_2| \) (respectively, \(|R^n_2(r)| \)).

We gather in Proposition 4 some simple observations.

**Proposition 4** Let \( d \) divide \( M \); say \( dk = M = n\ell \).

1. \( \gcd(d, \ell) = 1 \iff \text{lcm}(n, k) = M \).
2. If \( \gcd(d, \ell) = 1 \), then \( n/d = k/\ell = \gcd(n, k) \).
3. If \( \gcd(d, \ell) = 1 \), then for \( x_1 \neq x_2 \) in \( \mathbb{Z}_M \), \( x \equiv y \pmod{k} \Rightarrow x \neq y \pmod{n} \).

**Proof.**

1. In fact, more can be said:
\[
\gcd(d, \ell) \cdot \text{lcm}(k, n) = M.
\]
This follows from the observations
\[
x|d \Leftrightarrow k|(M/x) \text{ and } x|\ell \Leftrightarrow n|(M/x).
\]

2. Obviously \( n/d = k/\ell \); and \( kn/\gcd(k, n) = \text{lcm}(k, n) = M = \ell n \), so \( k/\gcd(k, n) = \ell \).

3. For \( x_1, x_2 \) in \( \mathbb{Z}_M \), if \( x_1 \equiv x_2 \pmod{k} \) and \( x_1 \equiv x_2 \pmod{n} \), then \( x_1 \equiv x_2 \pmod{\text{lcm}(k, n) = M} \), and therefore \( x_1 = x_2 \).

4.1 The total number of \( \ell \)-asymmetric rhythm cycles

**Theorem 5** The number of \( \ell \)-asymmetric rhythm cycles of length \( M = \ell n \) is

\[
|R^n_\ell| = \frac{1}{M} \left[ \sum_{d|M, \gcd(d, \ell) > 1} \phi(d) + \sum_{d|M, \gcd(d, \ell) = 1} \phi(d)(\ell + 1)^{n/d} \right]. \tag{4}
\]

**Proof.** With the group \( \mathbb{Z}_M \) acting on the set \( S^n_\ell \), by Burnside’s lemma, the number of orbits is \((1/M) \sum_{\beta \in \mathbb{Z}_M} \text{fix}(\beta)\), where \( \text{fix}(\beta) = |\{f \in S^n_\ell : \beta \cdot f = f\}| \). We note first that for each divisor \( d \) of \( M \), the elements of order \( d \) are precisely the generators of the \( d \)-element subgroup \( k\mathbb{Z}/M\mathbb{Z} \) of \( \mathbb{Z}_M \), where \( kd = M \); these are the \( \phi(d) \) elements \( \beta = kj \), where \( 1 \leq j \leq d \) and \( \gcd(j, d) = 1 \). Moreover, for each such \( \beta \), \( \beta \cdot f = f \) if and only if \( f \) is constant on each set \([x]_k\) (that is, \( f \) has period \( k \)).

Two cases now arise:

**Case 1.** If \( \gcd(d, \ell) > 1 \), so \( \text{lcm}(k, n) < M \), then for any \( x \in \mathbb{Z}_M \), in order for \( \beta \cdot f = f \), we must have \( f(x) = f(x + \text{lcm}(k, n)) \). Since \( x \neq x + \text{lcm}(k, n) \pmod{M} \), and since \( f \in S^n_\ell \), \( f \) must map both of these two elements to 0. So \( \beta \cdot f = f \) if and only if \( f(x) \equiv 0 \) for all \( x \). In this case, then, \( \text{fix}(\beta) = 1 \).
Case 2. Let \( \gcd(d, \ell) = 1 \). By Proposition 4 (3), the elements of each set \([x]_k\) are pairwise incongruent modulo \( n \), so that it is possible for a function \( f \in S^n\ell \) to map an entire class \([x]_k\) to 1. We next need to know, given two sets \([x_1]_k \neq [x_2]_k\), whether there exist \( y_1 \in [x_1]_k \) and \( y_2 \in [x_2]_k \) with \( y_1 \equiv y_2 \pmod{n} \); and it is not hard to see that this occurs if and only if \( x_1 \equiv x_2 \pmod{\gcd(n, k)} \). Accordingly, we partition \( \mathbb{Z}_M \) by congruence modulo \( g := \gcd(n, k) \) and examine the equivalence classes \([x]_g\), where \( 0 \leq x \leq (g - 1) \). Each class contains \( M/g = d\ell \) elements and is in fact a union of \( \ell \) of the sets \([x]_k\), and a function \( f : \mathbb{Z}_M \to \{0, 1\} \) of period \( k \) will be in \( S^n\ell \) if and only if, for each class \([x]_g\), \( f \) does one of \((\ell + 1)\) things: either it maps the entire class to 0, or it maps one of the subsets \([x]_k \subset [x]_g\) to 1 while mapping the rest of \([x]_g\) to 0. Constructing a function \( f \in S^n\ell \) fixed by the elements of order \( d \) thus entails making one of \((\ell + 1)\) choices independently for each of the classes \([x]_g\), so that in Case 2, \( \text{fix}(\beta) = (\ell + 1)^g = (\ell + 1)^{n/d} \).

Putting the two cases together now yields the result:

\[
|R^n\ell| = \frac{1}{M} \sum_{\beta \in \mathbb{Z}_M} \text{fix}(\beta) = \frac{1}{M} \left[ \sum_{d|\gcd(d, \ell)} \phi(d) + \sum_{d|\gcd(d, \ell) = 1} \phi(d)(\ell + 1)^{n/d} \right].
\]

Setting \( \ell = 2 \) in Equation 4 establishes Theorem 1.

Since the elements of \( S^n\ell \) may be thought of as \( \ell \)-asymmetric strings of length \( M \), the proof of Theorem 5 in fact gives the conditions under which an \( \ell \)-asymmetric string can have a period shorter than \( M \). Fix any divisor \( d \) of \( M \) and let \( \beta \in \mathbb{Z}_M \) be an element of order \( d \). Let \( f : \mathbb{Z}_M \to \{0, 1\} \) be any function, \( \ell \)-asymmetric or not, fixed by \( \beta \). Then \( f \) is periodic with period \( k = M/d \). Let \( \tilde{f} : \mathbb{Z}_k \to \{0, 1\} \) be the natural restriction of \( f \) to \( \mathbb{Z}_k \)—that is, for \( x = 0, \ldots, (k - 1) \), let \( \tilde{f}(x) = f(x) \).

**Corollary 6** Let \( f, \tilde{f}, d, \) and \( k \) be as above.

1. If \( f \) is not identically zero, then \( \gcd(d, \ell) = 1 \).

2. A function \( f \) is in \( S^n\ell \) if and only if \( \tilde{f} \) is in \( S^{k/\ell} \).

**Proof.**

1. This is exactly what is proved in Case 1 of Theorem 5.
2. Let $f$ be fixed by $\beta$, so that $f$ is determined by its values on $S = \{0, 1, \ldots, k-1\} \subseteq \mathbb{Z}_M$. If $\gcd(d, \ell) > 1$, then $f$ and $\tilde{f}$ are both identically zero, so that $f \in S^n_\ell$ and $\tilde{f} \in S_{k/\ell}^n$. Now let $\gcd(d, \ell) = 1$. As shown in the proof of Theorem 5, $f$ is in $S^n_\ell$ if and only if, for each class $[x]_g = [x]_{k/\ell}$, either $f$ maps the entire class to 0, or else $f$ maps exactly one subset $[x]_k \subseteq [x]_{k/\ell}$ to 1 while mapping the rest of $[x]_{k/\ell}$ to 0. Now, $S$ is a complete set of residues modulo $k$, so $f$ is in $S^n_\ell$ if and only if, for all $0 \leq x \leq (k/\ell - 1)$, either $f$ maps all of the set $[x]_{k/\ell} \cap S$ to 0, or $f$ maps exactly one element of this set to 1. But if one identifies $S$ with $\mathbb{Z}_k$, this becomes precisely the condition $\tilde{f}$ must satisfy to be in $S_{k/\ell}^n$.

Thus, when $\gcd(d, \ell) = 1$, elements of $S^n_\ell$ that are fixed by $\beta$ of order $d$ are in bijection with elements of $S_{k/\ell}^n = S^n_{\ell^d}$. Since $|S^n_{\ell^d}| = (\ell + 1)^n/d$, Corollary 6 gives a second perspective on the equation $\text{fix}(\beta) = (\ell + 1)^n/d$. We also see that $\ell$-asymmetric rhythms that are fixed by $\beta$ have a nice self-similarity property: they consist of $d$ copies of a rhythm that is $\ell$-asymmetric modulo $k$.

4.1.1 Example

Let $\ell = 5$ and $n = 6$. Corollary 6 tells us that any 5-asymmetric rhythm that is fixed by an element of order $d = 3$ consists of 3 copies of a rhythm that is 5-asymmetric modulo $k = 10$. Here is one such rhythm, consisting of 3 copies of the 5-asymmetric string $x\ldots x\ldots x\ldots x\ldots x\ldots x$,

with divisions of the measure into 5 parts shown:

Audio CD, Track 7

In Section 2.2, we gave two definitions of syncopation: (1) a temporary change in the division of a measure, and (2) a contradiction of the normal placement of strong and weak beats. Our definition of $\ell$-asymmetry follows (2). However, this rhythm, and indeed all $\ell$-asymmetric rhythms fixed by an element of order $d > 1$, are also syncopated in the sense of definition (1). A musician would think of this rhythm as “three against five,” because it involves the division of a measure in quintuple meter into three parts. This type of asymmetry is common in Central African music [4, p. 245].
4.2 The number of $r$-note $\ell$-asymmetric rhythm cycles

Now consider $S^n_\ell(r)$. Recall that $R^n_\ell(r) = S^n_\ell(r)/\mathbb{Z}_M$. Clearly, $|R^n_\ell(0)| = 1$.

**Theorem 7** For $1 \leq r \leq n$, the number of $\ell$-asymmetric rhythm cycles of length $M = tn$ with $r$ onsets is given by

$$|R^n_\ell(r)| = \frac{1}{M} \sum_{d \mid \text{gcd}(n,r)} \phi(d) \left( \frac{n/d}{r/d} \right) \ell^{r/d}.$$  \hspace{1cm} (5)

**Proof.** In outline, the proof is similar to that of Theorem 5. In this case, $|R^n_\ell(r)| = (1/M) \sum_{\beta \in \mathbb{Z}_M} \text{fix}_\ell(\beta)$, where $\text{fix}_\ell(\beta) := |\{ f \in S^n_\ell(r) : \beta \cdot f = f \}|$. Choose a divisor $d$ of $M$, put $k = M/d$, and let $\beta \in \mathbb{Z}_M$ be of order $d$. For any $f \in S^n_\ell(r)$, as before, $\beta \cdot f = f$ if and only if $f$ has period $k$. In the present context, though, there are three cases not two:

**Case 1.** Say $\text{gcd}(d, \ell) > 1$, so $\text{lcm}(k,n) < M$. The same argument as in Theorem 5, Case 1, shows that $\beta \cdot f = f$ only if $f(x) \equiv 0$ for all $x$. Since this is impossible for $f \in S^n_\ell(r)$, $\text{fix}(\beta) = 0$ in this case.

**Case 2.** Say $\text{gcd}(d, \ell) = 1$, but $d$ does not divide $r$. In order for $f$ to be constant on the orbits of $\beta$, necessarily $|f^{-1}\{1\}|$ must be a multiple of $d$. Since $d$ does not divide $r$, no function in $S^n_\ell(r)$ does this. Thus in Case 2 also, $\text{fix}(\beta) = 0$.

**Case 3.** Say $\text{gcd}(d, \ell) = 1$ and $d|r$. Then, for reasons set out in the proof of Theorem 5, Case 2, each function $f : \mathbb{Z}_M \to \{0,1\}$ satisfying the three conditions

1. $f$ is $\ell$-asymmetric
2. $|f^{-1}\{1\}| = r$
3. $x \equiv y \pmod{k} \Rightarrow f(x) = f(y)$

can be uniquely constructed as follows.

**Step 1.** Choose $r/d$ of the $g := \text{gcd}(k,n) = n/d$ classes $[x]_g$.

**Step 2.** For each of these $r/d$ classes, choose one of the $\ell$ classes $[x]_k$ contained in $[x]_g$.

**Step 3.** Let

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is in one of the chosen classes } [x]_k \\ 0 & \text{otherwise} \end{cases}$$
Clearly, then, the number of such functions is \( \left( \frac{n/d}{r/d} \right)^{\ell^{r/d}} \). Thus in Case 3, \( \text{fix}_r(\beta) = \left( \frac{n/d}{r/d} \right)^{\ell^{r/d}} \).

Taken together, the three cases now give

\[
|R^\beta_n(r)| = \frac{1}{M} \sum_{\beta \in \mathbb{Z}_M} \text{fix}_r(\beta) = \frac{1}{M} \sum_{d \mid \gcd(n,r)} \sum_{\gcd(d,\ell) = 1} \phi(d) \left( \frac{n/d}{r/d} \right)^{\ell^{r/d}}.
\]

Setting \( \ell = 2 \) in Equation 5 establishes Theorem 2.

4.2.1 Example

We now list the rhythm cycles of length 12 which are 3-asymmetric and have four note onsets. Using our formula for the number of \( r \)-note rhythm cycles, where \( \ell = 3 \) and \( r = n = 4 \), we see that there must be eight such cycles; we list all eight below.

1. |: xxxx........... :| 4. |: xx.x........x. :| 7. |: x.x...x.x.... :|
2. |: xxx....x.... :| 5. |: xx.x...x..... :| 8. |: x.x..x..x.. :|
3. |: xx....xx.... :| 6. |: x.....x..x.x :|

Notice that Patterns 3 and 8 are not primitive, meaning that they can be realized using a smaller period (Pattern 3 has primitive period 6, and Pattern 8 has primitive period 3). Patterns 5 and 6 are inversions of each other (that is, Pattern 5 is Pattern 6 played backwards); all other patterns are symmetric with respect to inversion.

4.3 Primitive \( \ell \)-asymmetric rhythm cycles

The previous example suggests that we also direct our attention to primitive rhythm cycles. A rhythm pattern of period \( M \) is primitive if it is not fixed by any nonidentity element \( \beta \neq \epsilon \) in \( \mathbb{Z}_M \); similarly, we deem a rhythm cycle primitive if the patterns in it are primitive. We wish to count primitive \( \ell \)-asymmetric rhythm cycles. Let \( P^n_\ell \) (respectively \( P^n_\ell (r) \)) denote the set of primitive cycles in \( R^n_\ell \) (respectively \( R^n_\ell (r) \)). Observe that each cycle in \( P^n_\ell \) (respectively \( P^n_\ell (r) \)) contains exactly \( M \) patterns, namely the \( M \) distinct cyclic shifts of any pattern in the cycle. Thus, the number of \( \ell \)-asymmetric rhythm patterns of period \( M \) (respectively, the number with \( r \) onsets) is given by \( M \cdot |P^n_\ell| \) (respectively \( M \cdot |P^n_\ell (r)| \)). We
make use of the Möbius function; recall that this function is defined on the positive integers by

\[
\mu(x) := \begin{cases} 
1 & \text{if } x = 1 \\
1 & \text{if } x \text{ is the product of an even number of distinct primes} \\
-1 & \text{if } x \text{ is the product of an odd number of distinct primes} \\
0 & \text{if } x \text{ is divisible by the square of any prime}
\end{cases}
\]

We also need the following notation: for integers \( n, \ell \geq 2 \), let \( \text{PR}_{n\setminus\ell} \) denote the set of primes that divide \( n \) but do not divide \( \ell \).

**Theorem 8** Let \( n, \ell, M, \) and \( \text{PR}_{n\setminus\ell} \) be as above.

1. If \( \text{PR}_{n\setminus\ell} = \emptyset \), then

\[
|P^n_\ell| = \frac{(\ell + 1)^n - 1}{M}.
\]

2. If \( \text{PR}_{n\setminus\ell} \neq \emptyset \), then

\[
|P^n_\ell| = \frac{1}{M} \left[ \sum_{\substack{d|n \\text{gcd}(d,\ell) = 1}} \mu(d)(\ell + 1)^{n/d} \right].
\]

**Proof.**

1. Consider any nonidentity rotation \( \beta \neq \epsilon \) in \( \mathbb{Z}_M \). Since \( \text{PR}_{n\setminus\ell} = \emptyset \), necessarily \( \text{gcd}(|\beta|,\ell) > 1 \), so by Corollary 6, \( \beta \) fixes only the zero function \( f_0(x) \equiv 0 \). Thus, every pattern in \( S^n_\ell \) except \( f_0 \) is primitive, so that

\[
M \cdot |P^n_\ell| = \text{the number of primitive patterns in } S^n_\ell = (\ell + 1)^n - 1.
\]

2. Let \( \text{PR}_{n\setminus\ell} \) comprise the distinct primes \( \{p_1, \ldots, p_t\} \). In this case, since there exist rotations \( \beta \neq \epsilon \) in \( \mathbb{Z}_M \) with \( \text{gcd}(|\beta|,\ell) = 1 \), we can say

\[
f \in S^n_\ell \text{ is primitive } \iff \beta \cdot f \neq f \text{ for all } \beta \neq \epsilon \text{ such that } \text{gcd}(|\beta|,\ell) = 1.
\]

Furthermore, \( f \) is fixed by a rotation of order \( d \) only if it is fixed by all rotations whose orders divide \( d \). It follows that we can restrict our attention to rotations whose order is prime:

\[
f \in S^n_\ell \text{ is primitive } \iff \beta \cdot f \neq f \text{ for all } \beta \text{ such that } |\beta| \in \text{PR}_{n\setminus\ell}.
\]
We now apply the Principle of Inclusion-Exclusion (PIE) (see [5, p. 124] for a precise statement and proof of the PIE). For each subset $T = \{p_1, \ldots, p_w\} \subseteq PR_{n \setminus \ell}$, we set

$$\text{FIX}(T) := \{ f \in S_n^\ell : |\beta| \in T \implies \beta \cdot f = f \}.$$

A moment's reflection shows that $f \in \text{FIX}(T)$ if and only if $f$ is fixed by the rotations $\beta$ of order $d = |\beta| = \prod_{j=1}^{w} p_{i_j}$, so that

$$|\text{FIX}(T)| = (\ell + 1)^{n/d}.$$

Applying the PIE now gives

$$M \cdot |P_\ell^n| = \sum_{T \subseteq PR_{n \setminus \ell}} (-1)^{|T|} |\text{FIX}(T)|$$

$$= (\ell + 1)^n - \sum_{1 \leq i \leq n} (\ell + 1)^{n/p_i} + \sum_{1 \leq i < j \leq n} (\ell + 1)^{n/p_ip_j} - \cdots$$

$$\cdots + (-1)^t(\ell + 1)^{n/p_ip_2 \cdots p_t}$$

$$= \sum_{d | n \atop \gcd(d,\ell) = 1} \mu(d)(\ell + 1)^{n/d}.$$

The case of $r \geq 1$ onsets is argued in completely analogous fashion. This time, happily, the case $PR_{n \setminus \ell} = \emptyset$ does not require a separate argument.

**Theorem 9** For $1 \leq r \leq n$, the number of primitive $\ell$-asymmetric rhythm cycles of length $M = \ell n$ with $r$ onsets is given by

$$|P_\ell^n(r)| = \frac{1}{M} \sum_{d | n \atop \gcd(d,\ell) = 1} \mu(d) \left( \frac{n/d}{r/d} \right) \ell^{r/d}. \quad (8)$$

We note that $|P_\ell^n(n)|$, which is closely related to $|R_\ell^n(n)|$ (given in Corollary 3), is sequence A000048 in [15].

**Alternate proof by Möbius inversion** It is also possible to prove Theorems 8 and 9 by Möbius inversion; the alternate proof of Theorem 8 runs as follows. Let $x$ be the product (counting multiplicities) of all primes that divide $n$ but do not divide $\ell$; in other words, if
\( \ell = q_1^{a_1} \cdots q_s^{a_s} \) and \( n = p_1^{b_1} \cdots p_s^{b_s} p_1^{c_1} \cdots p_t^{c_t} \), where the \( p \)'s and \( q \)'s are distinct primes and the \( a \)'s and \( c \)'s are all positive, then \( x = p_1^{c_1} \cdots p_t^{c_t} \) (if \( \text{PR}_{n|\ell} = \emptyset \), then \( x = 1 \)). Then, clearly, \( d|n \) and \( \gcd(d, \ell) = 1 \) if and only if \( d|x \). Now consider \( S_{\ell}^n \). By Corollary 6, each nonzero string \( f \in S_{\ell}^n \) is \( d \) copies of a primitive string in \( f_d \in S_{\ell|d}^{n/d} \) for a unique divisor \( d \) of \( x \) and a unique primitive function \( f_d \in S_{\ell|d}^{n/d} \), so

\[
(\ell + 1)^n - 1 = \sum_{d|n} \frac{M}{d} \left| P_{\ell|d}^{n/d} \right|.
\]

Furthermore, for each divisor \( d \) of \( x \), by the same argument applied to \( S_{\ell|d}^{n/d} \),

\[
(\ell + 1)^{n/d} - 1 = \sum_{d'|d} \frac{M/d}{d'} \left| P_{\ell|d'}^{(n/d)/d'} \right|.
\]

Then, by Môbius inversion (over the divisors of \( x \)),

\[
M \cdot |P_{\ell|n}^n| = \sum_{d|x} \mu(d) \left[ (\ell + 1)^{n/d} - 1 \right] = \sum_{d|n, \gcd(d,\ell) = 1} \mu(d) \left[ (\ell + 1)^{n/d} - 1 \right].
\]

This formula unifies formulas (6) and (7) in Theorem 8. It is equivalent to (7) when \( \text{PR}_{n|\ell} \neq \emptyset \) because for any \( x > 1 \), \( \sum_{d|x} \mu(d) = 0 \).

## 5 Applications to rhythmic canons

Our results on \( \ell \)-asymmetry have some surprising applications to certain types of rhythmic canons.

A canon is a musical figure produced when two or more voices play the same melody, with each voice coming in at a different time. Simple canons are called rounds; popular rounds include “Frère Jacques” and “Row, row, row your boat.”

We consider rhythmic canons—that is, canons in which rhythms, and not necessarily pitches, are duplicated by each voice. A rhythmic canon is produced when each voice plays a rhythm pattern, called the inner rhythm, and the voices are offset by amounts determined by a second pattern called the outer rhythm. We assume that both inner and outer rhythms may be expressed in terms of the same unit beat. For example, the inner rhythm \( xx...x..x..x \) together with the outer rhythm \( e.e..e \) (where \( e \)'s indicate voice entries) produces the three-voice rhythmic canon:

Audio CD, Track 16
In this notation, vertical alignment corresponds to alignment in time. Rhythmic canons were first studied mathematically by Vuza [16, 18, 17, 19]; see [2, 3, 1, 8] for further background.

5.1 Complementary canons

The French composer Olivier Messiaen (1908–1992), who coined the term “rhythmic canon,” used rhythmic canons in his work (Harawi, “Adieu,” and Visions de l’Amen, “Amen des anges, des saints, du chant des oiseaux”). He describes the sound of a rhythmic canon as a sort of “organized chaos” [12, p. 46]. In both pieces, he uses the inner rhythm

\[ \text{x..x....x.......x....x..x...x..x......x..x...x.x.x..x....x..x.x.} \]

A rhythmic canon is called complementary if, on each beat, no more than one voice has a note onset. For the most part, the note onsets of the voices in Messiaen’s canon are not simultaneous. Asterisks mark deviations from this “rule.”

5.1.1 Example

Here is a 12-beat rhythmic canon with three equally-spaced voices generated by the inner rhythm \( |: x.....x.x.x :| \) and outer rhythm \( |: e...e...e... :| \).
Voice 3: \[ x \ldots | : \ldots x \ldots x \ldots x : \]

Entries: \[ e \ldots e \ldots e \ldots e \ldots | : e \ldots e \ldots e \ldots e : \]

The actual canon is contained between the repeat signs; the pattern to the left is included to show how the voice entries are staggered. Note that this canon is complementary. This is because, for each \( x \in \mathbb{Z}_{12} \), the positions \( x, x + 4, \) and \( x + 8 \) in the inner rhythm contain no more than one onset, where addition is done mod 12. In other words, the inner rhythm is a 3-asymmetric rhythm cycle of length 12. In general, a rhythmic canon of \( \ell \) voices, each spaced by \( n \) notes from the previous, is complementary if and only if it is \( \ell \)-asymmetric.

5.2 Rhythmic tiling canons

A **rhythmic tiling canon** is a canon of periodic rhythms that has exactly one note onset (in some voice) per unit beat. Rhythmic tiling canons were introduced as regular complementary canons by Vuza [16, 18, 17, 19]; see also Andreatta et al. [2, 3]. In order to satisfy the tiling condition when \( \ell = 3 \) and \( n = 4 \), for each \( x \in \mathbb{Z}_{12} \), the positions \( x, x + 4, \) and \( x + 8 \) in the inner rhythm must contain exactly one onset, where addition is done mod 12. Thus, the inner rhythm is necessarily a 4-note 3-asymmetric cycle. Example 5.1.1 is also a tiling canon. In general, a rhythm of period \( \ell n = M \) forms a tiling canon of \( \ell \) equally-spaced voices if and only if it is \( \ell \)-asymmetric and has \( n \) note onsets.

Our results enumerate complementary and tiling canons in which both the inner and outer rhythms are periodic and in which the outer rhythm has equally spaced onsets (so that each voice is offset from the previous one by a fixed number of beats). The condition of periodicity appears in [16, 18, 17, 19, 2, 3, 1, 8]. However, the condition that onsets in the outer rhythm be equally spaced is introduced in this article.

**Corollary 10** The number of \( M \)-periodic complementary canons of \( \ell \) equally-spaced voices equals \( |R^n_\ell| \) (see Equation 4). The number of such canons in which the inner rhythm has \( r \) notes is given by \( |R^n_\ell(r)| \) (see Equation 5). The number of \( M \)-periodic tiling canons of \( \ell \) equally-spaced voices is

\[
|R^n_\ell(n)| = \frac{1}{M} \sum_{d|n, \gcd(d,\ell)=1} \phi(d) \ell^{n/d}. \tag{9}
\]
Corollary 11 The number of primitive $M$-periodic complementary canons of $\ell$ equally-spaced voices equals $|P^n|_\ell$ (see Equations 6 and 7). The number of such canons in which the inner rhythm has $r$ notes is given by $|P^n_r|$ (see Equation 8). The number of primitive $M$-periodic tiling canons of $\ell$ equally-spaced voices is

$$|P^n_{\ell}(n)| = \frac{1}{M} \sum_{d|n, \gcd(d,\ell)=1} \mu(d)\ell^{n/d}. \quad (10)$$

In Example 4.2.1, we listed the eight 3-asymmetric cycles of length 12 with four onsets; each of these determines a tiling canon. Audio recordings of the resulting canons are available at www.sju.edu/~rhall/Rhythms. It is interesting to listen to how the degree of symmetry affects the sound of the resulting canon; Patterns 5 and 6 sound to us the “most syncopated.”

5.3 Canons with unequal spacing of voices

Our condition that the voices in a canon be equally spaced greatly simplifies the problem of enumerating them. There exist rhythmic tiling canons with unequally-spaced voices. For example, the inner rhythm $\textcolor{red}{1}: \textcolor{blue}{x.x}\ldots :\textcolor{green}{1}$ and outer rhythm $\textcolor{red}{1}: \textcolor{blue}{ee}\ldots \textcolor{green}{:1}$ define an 8-periodic tiling canon. Frippertinger [9] has computed the numbers of tiling canons up to period 40. The question of finding equivalents of Corollary 10 and Corollary 11 for general rhythmic tiling canons is open. A rhythmic tiling canon in which both the inner and outer rhythms are primitive is called a canon of maximal category. Vuza showed that no nontrivial canons of maximal category exist for period of less than 72 [16, Theorem 2.2, p. 33]. We provide an audio example of a six-voice canon of maximal category and period 72. There is no known formula for the number of tiling canons of maximal category.

6 Other applications

6.1 Rhythmic oddity

Simha Arom [4, p. 246] pointed out that certain asymmetric rhythms played by peoples of the Central African Republic possess what he denotes the rhythmic oddity property, meaning that they are asymmetric rhythms with the additional restrictions that all note onsets are spaced by 2 or 3 units and that the period is $4n$, so that the rhythm splits into two patterns.
of length $2n - 1$ and $2n + 1$. The Afro-Cuban clave $\mid: x.x..x.x.x.. :\mid$ is one example. Chemillier [6] and Chemillier and Truchet [7] have developed an algorithm to generate all rhythms having the rhythmic oddity property. The question of a formula for the number of rhythms with the oddity property is still open.

6.2 One-dimensional tilings

Rhythmic tiling canons are, in fact, one-dimensional tilings of the integers. Our results on tiling canons also give the number of tilings of $\mathbb{Z}_M$ by equally-spaced tiles. There is extensive literature on this subject; for background, see [8].

References


