Cassiodorus (6th century)

Mathematics

- Arithmetic
- Music
- Geometry
- Astronomy

- Harmonics
- Metrics
- Rhythmics
Meter in Poetry

Much poetry follows one of a set of rhythmic rules, called *meters*.

In English and Spanish, the meter of a poem is determined by its pattern of stressed and unstressed syllables.

In Sanskrit (and many other languages), meter is determined by the pattern of long and short syllables. There are dozens of meters; some are determined by the number of syllables in a line, and some by the total length of the syllables in a line.

<table>
<thead>
<tr>
<th>Names of meters and numbers of syllables</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>uktā</em> 1</td>
</tr>
<tr>
<td><em>atyukta</em> 2</td>
</tr>
<tr>
<td><em>madhya</em> 3</td>
</tr>
</tbody>
</table>
Pingala’s Chandahsutra (c. 200 B.C.)

Syllables are short or long; in length

1 long = 2 short

Pingala (Bag, 1966) classified the 16 different meters of four syllables like this:

1 meter of four short syllables SSSS
4 meters of three shorts SSSL, SSLS, SLSS, LSSS
6 meters of two shorts LLSS, LSSL, SSLL, SLLS, LSL, SLLS, SLSL
4 meters of one short SLLL, LSLL, LLSS, LLLS
1 meter of no shorts LLLL

He described a very interesting pattern...
The Pattern...

- Purple = short
- Yellow = long

1 syllable

2 syllables

3 syllables

4 syllables
Pascal’s Triangle

North Africa, c. 1150
As Samaw’al ibn Yahya al-Maghribi

China, 1303
Chu Shih-Chieh

Germany, 1527
Petrus Apianus
Hemacandra enumerated meters in which the length of a line is fixed, but the number of syllables is not (Singh, 1986). Remember 1 long = 2 short. We’ll use squares for short syllables and dominoes for long syllables. What’s the pattern? Make a conjecture!

<table>
<thead>
<tr>
<th>patterns</th>
<th>num. of patterns</th>
</tr>
</thead>
<tbody>
<tr>
<td>length = 1</td>
<td>1</td>
</tr>
<tr>
<td>length = 2</td>
<td>2</td>
</tr>
<tr>
<td>length = 3</td>
<td>3</td>
</tr>
<tr>
<td>length = 4</td>
<td>5</td>
</tr>
<tr>
<td>length = 5</td>
<td>8</td>
</tr>
</tbody>
</table>
The Fibonacci Numbers

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ...

The first Fibonacci number is 1. After that, each number is the sum of the two previous numbers. A mathematician would express this relationship as

\[ F_N = F_{N-1} + F_{N-2} \]

where \( F_N \) is the \( N \)th Fibonacci number.

These numbers have an amazing variety of applications, not only to poetry, but also to nature, art, music, science, and many areas of mathematics.

See Ron Knott’s Fibonacci web site:

http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html
Got Proof?

Start with the patterns of length $N$ in one big pile

Separate them into two smaller piles:
- those ending in a long
- those ending in a short

OFF WITH the last syllables!
Drum Roll, Please...

There is 1 meter of length 1.
There are 2 meters of length 2.

For $N = 3, 4, 5, \ldots$, the number of meters of length $N$ is equal to

$$(\text{number of meters of length } N - 2) + (\text{number of meters of length } N - 1).$$

This is precisely the Fibonacci relationship!
Musical Rhythm

Rhythms in music are patterns of note onsets or accents. A *note* is the interval between successive attacks. It plays the role of a syllable in poetry.

**Rhythms of 1 and 2 beat notes**
- Merengue bell part (Dominican Rep.)
- Cumbia bell part (Columbia)
- Mambo bell part (Cuba)
- Bintin bell pattern (Ghana)
  - also Bembe Shango (Afro-Cuban)

**Rhythms of 2 and 3 beat notes**
- Lesnoto (Bulgaria)
- Bomba (Puerto Rico)
- Guajira (Spain)

Challenge

In Eastern European music, rhythms are composed of long and short notes, where a long note is 3 unit pulses and a short note is 2 unit pulses. For example, the Bulgarian lesnoto rhythm is a 7-beat rhythm of a long note followed by 2 short notes, written $3 + 2 + 2$, while the rachenitsa rhythm is $2 + 2 + 3$.

What sequence of numbers gives us the number of rhythms of length $N$?
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What sequence of numbers gives us the number of rhythms of length $N$?

<table>
<thead>
<tr>
<th>length ($N$)</th>
<th>1 2 3 4 5 6 7 8 9 10 11 12</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of patterns ($R_N$)</td>
<td>0 1 1 1 2 2 3 4 5 7 9 12</td>
</tr>
</tbody>
</table>

What’s the next number? What’s the rule? Can you prove it?
Padovan Numbers (Stewart, 2004)

The Padovan sequence has the same relationship to a spiral of equilateral triangles that the Fibonacci sequence has to a spiral of squares.

The limit of the ratios of successive terms in the Padovan sequence is the real solution to \( r^3 - r - 1 = 0 \), or approximately \( r = 1.3247179572447 \ldots \), the *plastic number*. 
Divisive Rhythm

Divisive rhythm occurs when the basic unit of time (a measure) is subdivided into a number of equal notes, each of which may be further subdivided.

Here’s the classic picture. In this case, we’re only dividing notes into halves—we’ll call this *binary division type*. [Listen](#)
Syncopation

The bomba rhythm (represented above) is perceived as a deviation from the expected pattern of accents. This deviation creates a rhythmic tension known as *syncopation*. Syncopated rhythm is common in African-American and Latin American music. It is also highly developed in Renaissance music.

**Examples:** (1) E. T. Mensah, Ghanaian pop song [Listen](#) (2) Cuarteto Patria, “Chan-chan” [Listen](#)

*How many syncopated rhythm patterns of k notes are there?*
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*How many syncopated rhythm patterns of \(k\) notes are there?*

The difficulty lies in defining syncopation mathematically. Perhaps we can define “not syncopated”...
Uniform Rhythm Patterns

Uniform rhythms are definitely not syncopated. Example: (1) The Beatles, “Free as a Bird” (2) MIDI

Let a *uniform rhythm* be any rhythm pattern that results from a sequence of equal divisions of notes. For example, \[\frac{4}{1} \frac{1}{1} \frac{1}{2}\] is a uniform rhythm. The corresponding sequence would be

1. 
2. 
3. 
4. 

However, \[\frac{3}{1} \frac{1}{1} \frac{1}{1} \frac{2}{1}\] is not uniform, since the first half of the measure has been divided in the ratio \(3 : 1\).
The Catalan numbers got rhythm!

**Theorem 1 (Hall and Klingsberg)** *The number of uniform rhythm patterns of binary division type with \( k \) notes equals the Catalan number \( C(k - 1) = (2k - 2)!/(k!(k - 1)!) \).*

Catalan numbers: 1, 1, 2, 5, 14, 42, 132, 429, 1430,\ldots
Are “Non-uniform” and “Syncopated” Equivalent?

All uniform rhythms are non-syncopated. However, a musician would probably not agree that all non-syncopated rhythms are uniform. For example, \[ \begin{array}{cccc}
3 & 1 & 1 & 1 \\
\end{array} \begin{array}{c}
2 \\
\end{array} \] does not sound syncopated.

According to the *New Harvard Dictionary of Music*, syncopation is “...a momentary contradiction of the prevailing meter or pulse. This may take the form of a temporary transformation of the fundamental character of the meter ...or it may be simply the contradiction of the regular succession of strong and weak beats within a measure or group of measures (Randel, 1986).”

So this example is not syncopated because long notes correspond to strong beats.
Asymmetric Rhythms

Paul Klingsberg and I have studied asymmetric rhythms, which are commonly found in African drumming. They are rhythms that are “maximally syncopated” in that, even when delayed, they never are aligned with the primary divisions in a measure. Example:

Examples: (1) Aka Pygmies, “Bobangi”  (2) MIDI

Our results had surprising applications to the study of rhythmic canons…

For more information, see
http://www.sju.edu/~rhall/research.html
In the classroom

Multicultural Math
http://www.sju.edu/~rhall/Multi
Abcdrums
http://www.sju.edu/~rhall/Multi/drums.html

Students

Mateo Quiñones
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Brian Centanni
Listen

Kate McCormick
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References


