Asymmetric Rhythms and Tiling Canons

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Rhythm patterns

We define rhythm patterns as sequences of attacks, accents, or changes in tone color. A *note* is the interval between successive attacks, also called *note onsets*.

We assume there is some invariant beat that cannot be divided, so that every note onset occurs at the beginning of a beat.

Notation

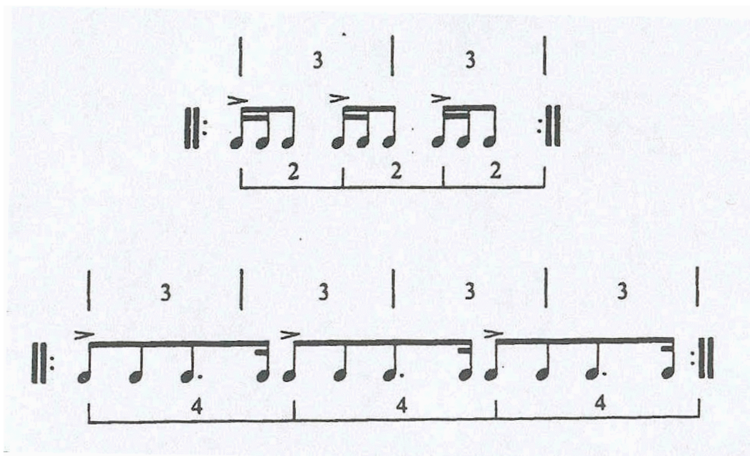
The following are equivalent.

standard	♩. ♩. ♩ or ♪𝄾 𝄾 ♪𝄾 𝄾 ♪𝄾
additive	$3 + 3 + 2$
drum tablature	`x..x..x.`
binary	10010010

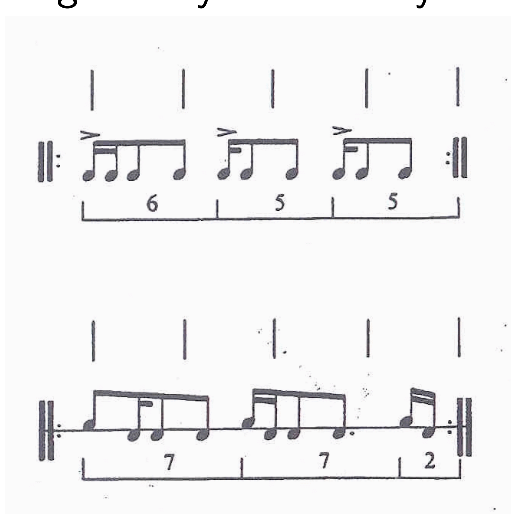
Asymmetric rhythm patterns

Asymmetric rhythm patterns cannot be broken into two parts of equal duration, where each part starts with a note onset. Arom (1991, p. 245-6) classifies many Central African rhythms as asymmetric.

Regular asymmetric rhythms

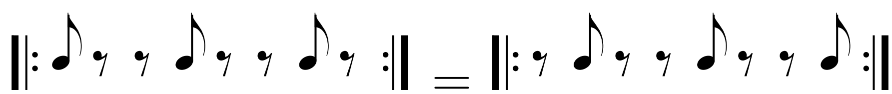


Irregular asymmetric rhythms



Rhythm cycles

We say that two periodic rhythms are equivalent if one is a shift of the other. For example,



where repeat signs indicate "infinite repeats."

We call the equivalence class consisting of all cyclic shifts of a rhythm pattern a *rhythm cycle*.

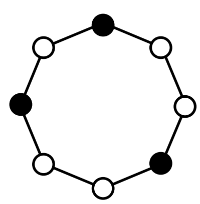
Binary necklaces

A *binary necklace* is an equivalence class of finite binary sequences, where two sequences are equivalent if one is a cyclic shift of the other:

$$0010011 \equiv 1001100$$

Binary necklaces are also represented as necklaces of black and white beads equivalent under rotation. Turning over a necklace (inversion) is not allowed.

Every rhythm cycle is represented by a unique binary necklace.

10010010 $\longleftrightarrow$  $\longleftrightarrow$ 

Many mathematical results on binary necklaces have interesting applications to rhythm cycles.

Asymmetric rhythm cycles

Asymmetric rhythm cycles are equivalence classes of asymmetric patterns—they cannot be delayed so that note onsets coincide with both of the strong beats in the measure. Many periodic rhythms from Africa are asymmetric cycles (Arom, 1991; Chemillier, 2002).

The rhythm cycle |: `x..x..x.` :| is asymmetric.

However, the cycle |: `x.x...x.` :| is not, as

$$|:\ \texttt{x.x...x.}:| = |:\ \texttt{x...x.x.}\ :|$$

Rhythmic oddity

Asymmetric rhythm cycles composed only of notes of length 2 and 3 are of particular significance in Arom's study. The question of counting cycles with this property has been answered by Chemillier and Truchet (2003).

Rhythms as functions

A rhythm pattern is represented by a function

$$f : \mathbb{Z} \to \{0, 1\},$$

where

$$f(x) = \begin{cases} 1 & \text{if there is a note onset on pulse } x \\ 0 & \text{otherwise} \end{cases}$$

A periodic function with period p can be identified with a function

$$f : \mathbb{Z}_p \to \{0, 1\}.$$

A *rhythm cycle* is an equivalence class of functions on $\mathbb{Z}_p$ modulo rotation. That is, f_1 is equivalent to f_2 if $f_1(x) = f_2(x - k)$ holds for some k and for all x.

The asymmetry condition

A rhythm pattern of period $p = 2n$ is *asymmetric* if

$$f(x) = 1 \quad \Longrightarrow \quad f(x + n) = 0$$

for all $x \in \mathbb{Z}_{2n}$.

Counting asymmetric rhythm patterns

Note that asymmetric patterns only occur in duple meter.

Let $S_2^n = \{$asymmetric patterns of length $2n\}$.

Then the size of S_2^n is 3^n (written $|S_2^n| = 3^n$).

Partition the elements of $\mathbb{Z}_{2n}$ *into* n *pairs* $\{\{0, n\}, \ldots, \{n-1, 2n-1\}\}$*; choose either zero or one element of each pair to be mapped to 1.*

Let $S_2^n(r) \subset S_2^n$ be the subset of asymmetric patterns with r note onsets $(0 \leq r \leq n)$.

Then $|S_2^n(r)| = \binom{n}{r} 2^r$.

Choose r *of the pairs above; choose one element of each pair to be mapped to 1.*

Let $P_2^n \subset S_2^n$ and $P_2^n(r) \subset S_2^n(r)$ represent *primitive* asymmetric patterns—those whose primitive period is $2n$. We also have formulas that count primitive rhythm patterns.

Burnside's Lemma

The cyclic group $\mathbb{Z}_{2n}$ acts on patterns of length $2n$ by cyclic shift.

Precisely, an element $m \in \mathbb{Z}_{2n}$ acts on a function f in S_2^n by shifting it through m positions.

For example, if f corresponds to 1001100 and $m = 1$, $m \cdot f$ corresponds to 0100110.

We'll count equivalence classes in each set of patterns modulo cyclic shifts. These classes are *orbits* induced by a group action. Therefore, we can apply Burnside's Lemma.

Burnside's Lemma *Let a finite group G act on a finite set S; for each $\beta \in G$, define $\mathrm{fix}(\beta)$ to be the number of elements s in S such that $\beta \cdot s = s$. Then the number of orbits that G induces on S is given by*

$$\frac{1}{|G|} \sum_{\beta \in G} \mathrm{fix}(\beta).$$

The number of asymmetric rhythm cycles

Theorem 1 *The number of asymmetric rhythm cycles of period* $2n$ *is*

$$\frac{1}{2n}\left[\sum_{d|n}\phi(2d) \ + \sum_{\substack{d|n \\ d \text{ odd}}} \phi(d)3^{n/d}\right],$$

where $\phi(d)$ *is Euler's totient function (i. e.,* $\phi(d)$ *is the number of integers* $1 \le x \le d$ *that are relatively prime to* d*).*

Proof. For each divisor d of $2n$, find the elements β of order d and determine $\operatorname{fix}(\beta)$, which depends only on d. Note that for each d, the number of β of order d is $\phi(d)$.

Case 1. d divides $2n$ and d is even.

The only asymmetric element preserved by a rotation of even order is the empty rhythm $f \equiv 0$, so $\mathrm{fix}(\beta) = 1$.

Case 2. d divides $2n$ and d is odd.

An element preserved by a rotation of odd order is asymmetric if $f(x) = 1 \Rightarrow f(x + n/d) = 0$. The elements $0, \ldots 2n/d$ of $\mathbb{Z}_{2n}$ are partitioned into n/d pairs $\{\{0, n/d\}, \ldots, \{n/d - 1, 2n/d - 1\}\}$; either zero or one element of each pair is sent to 1.

□

The number of r-note asymmetric rhythm cycles

Theorem 2 *If $1 \leq r \leq n$, then the number of asymmetric rhythm cycles with r note onsets is given by*

$$\frac{1}{2n} \sum_{\substack{d \mid \gcd(n,r) \\ d \text{ odd}}} \phi(d) \binom{n/d}{r/d} 2^{r/d}.$$

Proof. For each divisor d of $2n$, we will find the elements β of order d and determine $\text{fix}(\beta)$.

Case 1. d divides $2n$ and d is even.

Since $r \geq 1$, $\text{fix}(\beta) = 0$.

Case 2. d divides $2n$ and d is odd.

(a) d divides r

The elements $0, \ldots, 2n/d$ *of* $\mathbb{Z}_{2n}$ *are partitioned into* n/d *pairs* $\{\{0, n/d\}, \ldots, \{n/d - 1, 2n/d - 1\}\}$. *Choose* r/d *of these pairs and send one of the elements to 1.*

(b) d does not divide r

Since $r \geq 1$, $\text{fix}(\beta) = 0$.

□

We also have formulas that count primitive rhythm cycles.

Asymmetry in other meters

Asymmetry is defined by Arom with respect to duple meter. But a similar situation can occur in any meter...

Suppose we divide a measure of M beats into ℓ equal parts and place a strong beat at the beginning of each part, creating "ℓ-tuple meter."

Rhythm cycles are asymmetric in ℓ-tuple meter if, even when shifted, they cannot be broken into ℓ parts of equal duration, where more than one part starts with a note onset.

Definition of ℓ-asymmetry

We say that a periodic rhythm of period ℓn is *ℓ-asymmetric* if when position x contains a note onset, then all other positions y, where $y \equiv x \pmod{n}$, do not contain note onsets.

For example, the 12-periodic rhythm cycle

```
|: x.....x..x.x :|
```

is 3-asymmetric ($n = 4$). All of its shifts contain no more that one onset among the three principal beats in triple meter.

Note that our previous definition of asymmetry corresponds to ℓ-asymmetry when $\ell = 2$.

Counting ℓ-asymmetric rhythm cycles

Theorem 3 *The number of ℓ-asymmetric rhythm cycles of length $M = \ell n$ is*

$$\frac{1}{M}\left[\sum_{\substack{d|M\\ \gcd(d,\ell)>1}} \phi(d) \; + \sum_{\substack{d|n\\ \gcd(d,\ell)=1}} \phi(d)(\ell+1)^{n/d}\right].$$

The number of r-note ℓ-asymmetric rhythm cycles

Theorem 4 *For $1 \leq r \leq n$, the number of ℓ-asymmetric rhythm cycles of length $M = \ell n$ with r onsets is given by*

$$\frac{1}{M}\sum_{\substack{d|\gcd(n,r)\\ \gcd(d,\ell)=1}} \phi(d)\binom{n/d}{r/d}\ell^{r/d}.$$

Primitive ℓ-asymmetric rhythm cycles

In what follows μ signifies the classical Möbius function: $\mu(1) = 1$ and

$$\mu(d) = \begin{cases} 0 & \text{if } x \text{ is divisible by the square of any prime} \\ 1 & \text{if } x \text{ is the product of an even number of distinct primes} \\ -1 & \text{if } x \text{ is the product of an odd number of distinct primes} \end{cases}$$

Theorem 5 *The number of primitive ℓ-asymmetric rhythm cycles of length $M = \ell n$ is*

$$\frac{1}{M} \sum_{\substack{d|n \\ \gcd(d,\ell)=1}} \mu(d)[(\ell+1)^{n/d} - 1].$$

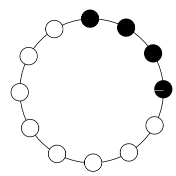
Theorem 6 *If $1 \leq r \leq n$, then the number of primitive ℓ-asymmetric rhythm cycles of length $M = \ell n$ with r onsets is given by*

$$\frac{1}{M} \sum_{\substack{d|\gcd(n,r) \\ \gcd(d,\ell)=1}} \mu(d) \binom{n/d}{r/d} \ell^{r/d}.$$

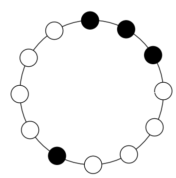
Example

We now list the rhythm cycles of length 12 which are 3-asymmetric and have four note onsets. ($\ell = 3$ and $r = n = 4$). There are eight:

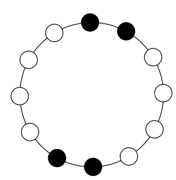
1. |: xxxx........ :|



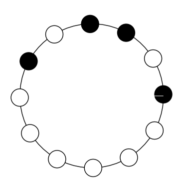
2. |: xxx....x.... :|



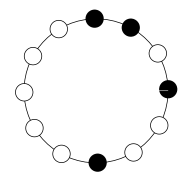
3. |: xx....xx.... :|



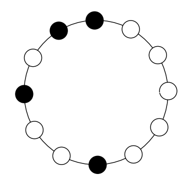
4. |: xx.x......x. :|



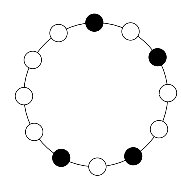
5. |: xx.x..x..... :|



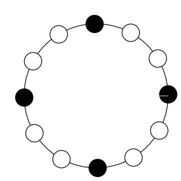
6. |: x.....x..x.x :|



7. |: x.x..x.x.... :|



8. |: x..x..x..x.. :|



Patterns 3 and 8 are not primitive.

Patterns 5 and 6 are inversions of each other; all other patterns are symmetric with respect to inversion.

Applications to rhythmic canons

Messaien (1992) coined the term *rhythmic canon*, which is produced when each voice plays a rhythm pattern (the *inner rhythm*), and the voices are offset by amounts determined by a second pattern (the *outer rhythm*).

The terms *inner rhythm* and *outer rhythm* come from (Andreatta et al., 2002).

Example

$$\begin{aligned} \text{inner rhythm} &= \texttt{x x . . x . x . x} \\ \text{outer rhythm} &= \texttt{e . e . . e} \end{aligned}$$

The canon:

```
Voice 1:  x x . . x . x . x
Voice 2:      x x . . x . x . x
Voice 3:            x x . . x . x . x

Entries:  e . e . . e
```

Complementary canons

Messiaen described the sound of a rhythmic canon as a sort of “organized chaos” (Messiaen, 1992, p. 46). In *Harawi*, “Adieu,” he uses the inner rhythm

```
x..x....x.......x....x..x...x..x......x..x...x.x.x..x....x
```

together with the outer rhythm `e.e.e` to generate the canon

```
V.1:  x..x....x.......x....x..x...x..x......x..x...x.x.x..x....x
V.2:    x..x....x.......x....x..x...x..x......x..x...x.x.x..x....x
V.3:      x..x....x.......x....x..x...x..x......x..x...x.x.x..x....x

Ent:  e.e.e
```

A rhythmic canon is called *complementary* if, on each beat, no more than one voice has a note onset.

Messiaen’s canon is *almost* complementary.

Canons of periodic rhythms

In this paper we study canons of periodic rhythms. This is consistent with (Vuza, 1991–1993; Andreatta et al., 2002). For example,

```
inner rhythm = |: x.....x....x :|
outer rhythm = |: e...e...e... :|.
```

defines a 12-beat rhythmic canon with three equally-spaced voices:

```
x . . . . . x . . . . . x |: x . . . . . . x . . . . . x :|
        x . . . . . . x . |: . . . x x . . . . . . x . :|
                x . . . . |: . . x . . . . . x x . . . . :|

e . . . e . . . e . . . . |: e . . . e . . . e . . . . :|
```

The inner rhythm is a 3-asymmetric rhythm cycle of length 12.

> A rhythmic canon of ℓ voices, each spaced by n notes from the previous, is complementary
>
> $\iff$
>
> its inner rhythm is ℓ-asymmetric.

Rhythmic tiling canons

A *rhythmic tiling canon* is a canon of periodic rhythms that has *exactly* one note onset per beat (Vuza, 1991–1993; Andreatta et al., 2002).

In order to satisfy the tiling condition when $\ell = 3$ and $n = 4$, for each $x \in \mathbb{Z}_{12}$, the positions x, $x + 4$, and $x + 8$ in the inner rhythm must contain exactly one onset. Thus, the inner rhythm must be a 4-note 3-asymmetric cycle.

> A rhythm of period $\ell n = M$ forms a tiling canon of ℓ equally-spaced voices
>
> $\Longleftrightarrow$
>
> it is ℓ-asymmetric and has n note onsets.

Example

Each of the eight 3-asymmetric cycles of length 12 with four onsets determines a tiling canon.

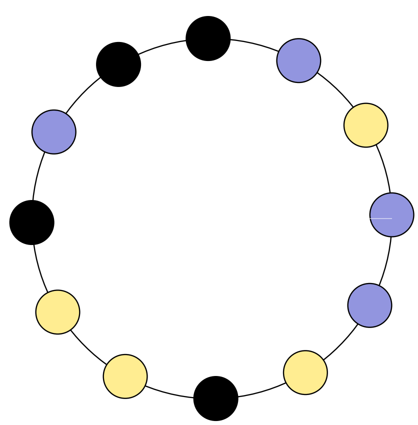
$$\begin{aligned} \text{inner rhythm} &= \texttt{|: x.....x..x.x :|} \\ \text{outer rhythm} &= \texttt{|: e...e...e... :|}. \end{aligned}$$

defines a 12-beat rhythmic canon with three equally-spaced voices:

```
x . . . . . x . . x . x |: x . . . . . x . . x . x :|
        x . . . . . x . |: . x . x x . . . . . x . :|
                x . . . |: . . x . . x . x x . . . :|

e . . . e . . . e . . . |: e . . . e . . . e . . . :|
```

As a necklace. . .



Problems

Tiling canons with unequally-spaced voices. For example,

$$\begin{aligned} \text{inner rhythm} &= \texttt{|: x.x..... :|} \\ \text{outer rhythm} &= \texttt{|: ee..ee.. :|.} \end{aligned}$$

How can we classify and count these canons?

Maximal category. Tiling canons of maximal category are tiling canons where both the inner rhythm and the outer rhythm are primitive. None exist for periods less than 72 (Vuza, 1991–1993).

Inversion. The problem of finding tiling canons using one rhythm and its inversion (see Beethoven Op. 59, no. 2—the patterns are `xx..x.` and `..xx.x`.) is equivalent to one studied by Meyerowitz (2001). He showed that any rhythm with three onsets must tile in this way, but the general question remains open. We have analogues of Theorems 4 and 6 for the cases in which the full dihedral group acts on the set of ℓ-asymmetric patterns.

Tilings of the integers. Rhythmic tiling canons are, in fact, one-dimensional tilings of the integers using a single tile. All such tilings are periodic (Newman, 1977). Our results on tiling canons give the number of tilings of $\mathbb{Z}$ by equally-spaced placements of a single tile. There are many open questions on one-dimensional tilings.

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