Many of the qualities that differentiate between random sounds and musical sounds involve periodicity. We discussed periodic phenomena in class, and you should be familiar with the terms periodic, fundamental cycle, fundamental period, frequency, tempo, Hz, and BPM.

Mathematicians use what are called periodic functions to represent sound waves. Here’s the mathematical definition of a function:

**Definition.** A function is a rule that assigns a unique output number to every input number. The input number (often denoted \( x \) or \( t \)) is called the independent variable and the output number (often denoted \( y \)) is the dependent variable. If \( f \) is a function whose input is \( x \) and whose output is \( y \), we write

\[
y = f(x).
\]

**Examples.**

We can represent any function by a graph. The graph of a function \( f \) is the set of all points \( (x, y) \) where \( y = f(x) \) (that is, \( y \) is the output when \( x \) is the input). We can draw the graph of \( f \) on the Cartesian plane with \( x \) (or \( t \)) on the horizontal axis and \( y \) on the vertical axis.

**Examples.**

When representing a vibration, the independent variable is time \( (t) \) and the dependent variable \( (y) \) is displacement. When the vibrating object is at rest, \( y = 0 \). Let’s graph the displacement of a point on a violin string when the player draws the bow across the string.
How can you tell if a mathematical function is periodic, other than by drawing its graph? Here is the definition:

**Definition.** Suppose \( f \) is a function and \( P \) is a positive number. If

\[
f(t) = f(t + P)
\]

for all possible values of \( t \) then \( f \) is a periodic function.

**Example.** Look back at the graph you drew of the violin string. List values of \( P \) where \( f(t) = f(t + P) \). There is more than one correct answer—in fact, there are infinitely many correct answers!

\[ P = \] 

**Example.** The floor function outputs the greatest integer (whole number) less than or equal to the input \( t \). On a TI calculator, the floor function is denoted \( \text{int} \). So \( \text{int}(5.729) = 5 \), \( \text{int}(1/2) = 0 \), \( \text{int}(4) = 4 \), and \( \text{int}(-2.45) = -3 \). (Careful with that last one!) Let’s define a function \( f \) like this:

\[
f(t) = t - \text{int}(t).
\]

For example,

\[
f(5.729) = 5.729 - \text{int}(5.729) = 5.729 - 5 = 0.729.
\]

Let’s make a table of input and output values.

Can you give examples of positive numbers \( P \) where \( f(t) = f(t + P) \), no matter what \( t \) is?

\[ P = \] 

Let’s sketch a graph of \( f \). You can also try this on a graphing calculator if you have one.

Suppose that \( t \) is measured in milliseconds (1 second = 1000 milliseconds). What is the fundamental period of \( f \)? The frequency? Give your answer in both Hz and BPM. Is a sound with this frequency audible?
The graphs that we have drawn suggest how we can define the fundamental period and frequency of a function mathematically:

The *fundamental period* of a periodic function \( f \) is the smallest positive number \( P \) for which the equation \( f(t) = f(t + P) \) is true for all \( t \), and the *frequency* of a periodic function equals \( 1/P \).

**Sinusoidal Functions**

**Important:** in this class, set your calculator in Radian mode, rather than Degree mode.

For our purposes, the *sinusoidal functions*, sine and cosine, are the most important examples of periodic functions. This is because they can be used to study how our ears hear sounds.

Here’s one way to define the sine function. The *unit circle* is the circle with radius of one unit and center at the origin on the Cartesian plane. Now imagine that an ant starts crawling around the circle in a counterclockwise direction, starting at the point \((1, 0)\). Suppose \( t \) represents the distance she has traveled on the circle and \( y \) is the \( y \)-coordinate of her location on the plane. Let’s make a table of values for \( t \) and \( y \), and sketch a graph.

The function that inputs \( t \) and outputs \( y \) is called the *sine function*, \( y = \sin t \). If you set your calculator in radian mode, you can find more values of the function. (You probably learned the sine function when studying trigonometry. The *radian* measure of an angle is the length of the arc it sweeps out on the unit circle. If \( t \) is an angle measured in radians, then the \( y \) value that we obtained is exactly \( \sin t \).)

What is the fundamental period of the sine function? The frequency?

**Exercise 5.** Imagine the same experiment with the ant, but this time make the input be distance \( (t) \) and the output be the \( x \)-coordinate of the ant’s position. Sketch the graph. What is the fundamental period? The frequency? What is the name for this function?
Variations on the Theme of Sinusoidal Functions

Let’s sketch some variations on the sine function.

1. \( y = 4 \sin t \)
2. \( y = 4 + \sin t \)
3. \( y = \sin(4t) \)
4. \( y = \sin(t/4) \)
5. \( y = \sin(4 + t) \)

In general, any function that can be written in the form \( f(t) = D + A \sin(Bt + C) \) where \( A > 0 \) and \( B > 0 \) is called a sinusoidal function. The numbers \( A, B, C, \) and \( D \) are called parameters. Their values determine how the sine function is shifted and stretched from its basic shape. The value of \( B \) is particularly important for our purposes. The fundamental period of \( f \) is \( 2\pi/B \), so the frequency of \( f \) is \( B/(2\pi) \). If we use the letter \( F \) to represent the frequency, then \( F = B/(2\pi) \), so \( B = 2\pi F \). Therefore, let’s use the following equation as the general form of a sinusoidal function:

\[
f(t) = D + A \sin(2\pi F t + C).
\]

In this form,

- The parameter \( F \) is the frequency of the sinusoid. If \( t \) is measured in seconds, then \( F \) is a value in Hertz. The fundamental period is \( 1/F \).
- The parameter \( A \) is the amplitude. The distance between the largest and smallest values of \( f(t) \) is exactly \( 2A \). If \( f \) is a sound wave, \( A \) determines its loudness.
- The parameter \( D \) is the average value or “center” of the wave in the vertical direction. (Normally, for our purposes, \( D = 0 \) because the average displacement should be 0.)
- The parameter \( C \) determines the horizontal shift, or phase, of the sinusoid. If \( f \) is a sound wave, changing the value of \( C \) does not affect its sound.