Sounding Number
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Spectrum and Timbre

The equation
\[ u(x, t) = \left( \sin \left( \frac{n\pi x}{L} \right) \right) \left( A \sin \left( \frac{an\pi t}{L} + C \right) \right), \]
where \( L \) is the length of the tube or string, \( a \) is a physical constant, and \( n \) is a positive whole number, is a solution to the wave equation (for a clarinet, \( n \) is an odd number). It predicts the frequencies at which the harmonics of a stringed or wind instrument will appear.

However, the characteristic sound of an instrument, called its timbre (TAM-ber) comes from the fact that the instrument actually produces all of its harmonics simultaneously. The lowest harmonics are the loudest and the ones that we identify as the fundamental frequency of the sound. Suppose we choose a fixed \( x \) (that is, we just look at what happens at one particular point on the tube). Remember that the fundamental frequency \( f_0 \) equals \( an/2L \). Then the vibration can be modeled by

\[ u(t) = A_1 \sin(2\pi f_0 + C_1) + A_2 \sin(4\pi f_0 + C_2) + A_3 \sin(6\pi f_0 + C_3) + A_4 \sin(8\pi f_0 + C_4) + \cdots. \]

The formula predicts that the vibration produced by the flute is a sum of sinusoids, each of which has frequency equal to some multiple of the fundamental frequency. The amplitudes \((A_1, A_2, A_3, \ldots)\) of the sinusoids vary; generally, the amplitudes get smaller as \( n \) gets larger. (It’s also true that the highest frequencies are not hearable by humans—if \( f_0 \) equals 440 Hz, then the fiftieth harmonic is 22,000 Hz, which is not audible.)

A representation of frequencies and their corresponding amplitudes is called a spectrum (plural: spectra). The most common way to display a spectrum is to make a plot with frequency (in Hz) on the horizontal axis and intensity (in decibels, or dB) on the vertical axis. (Intensity is determined by amplitude, and higher amplitude means higher intensity.)

Let’s take a look at what we see in recordings of actual instruments. The first recording I made is of a tuning fork. Here’s a picture of the sound wave:

![Sound Wave Image]

This looks like a sine wave! Let’s highlight a portion of the sound in Audacity and use Analyze → Spectrum to make a plot of the spectrum. There is a large spike at 440 Hz, a much smaller spike at 880 Hz, and not much else interesting.
For comparison, look at the sound and spectrum of a violin: