Wind Instruments and Harmonics

The Norwegian folk flute called the *seljefløyte*, or willow flute, is a member of the recorder family, though it is held crosswise. The flute is made from a hollow willow branch (more recently, a PVC pipe). One end is open and the other contains a slot into which the player blows, forcing air across a notch in the body of the flute. The resulting vibration creates standing waves inside the instrument whose frequency determines the pitch that we hear. The willow flute does not rely on finger holes to produce different notes. Rather, by varying the strength with which he or she blows into the flute, the player selects from a series of notes called *harmonics*. It is evident from the tune *Willow Dance*, as performed by Hans Brimi on the willow flute, that quite a number of different tones can be produced on the willow flute. How is this possible?

The answer lies in the mathematics of sound waves. The equations for sound may be found in most textbooks on Differential Equations, which is a branch of Calculus. Suppose $x$ is the position along the length of the tube and $t$ is time. The pressure at position $x$ and time $t$ is a function of both $x$ and $t$, so we write $u(x, t)$ for the pressure. (The pressure across the tube is close to constant, so we can ignore that direction.) We choose units such that the pressure outside the tube is zero. Since both ends of the tube are open, the pressure at the ends is zero. That is, if $L$ is the length of the tube, $u(0, t) = 0$ and $u(L, t) = 0$.

The one-dimensional wave equation $a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ provides a good model of the behavior of air molecules in the tube. This is called a *partial differential equation*. Here, $a$ is a positive number that is determined by the physical properties of air. Solutions to the wave equation are sums of solutions of the form

$$u(x, t) = (\sin Bx)(A \sin(aBt + C))$$

where $B > 0$. We can figure out the possible values of $B$ using trigonometry. The pressure must be zero at the ends of the tube (that is, at $x = 0$ and $x = L$). Since $\sin 0 = 0$, this is automatically true when $x = 0$. In addition,

$$\sin BL = 0, \quad \text{so } BL = \pi, 2\pi, 3\pi, \ldots.$$ 

Solving for $B$, we obtain

$$B = \frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L}, \ldots.$$ 

Therefore,

$$u(x, t) = \left(\sin \frac{n\pi x}{L}\right) \left(A \sin \frac{an\pi t}{L} + C\right)$$

where $n = 1, 2, 3, \ldots$, and $A$ is a constant number. Each value of $n$ represents a different *standing wave* in the tube.
How does our solution predict the possible frequencies of tones produced by the flute? Fix $n$ and $x$ and vary $t$. The pressure varies periodically with

$$\text{frequency} = \frac{B}{2\pi} = \frac{an\pi/L}{2\pi} = \frac{an}{2L} \quad \text{for} \quad n = 1, 2, 3, \ldots$$

This formula suggests that there are two ways to play a wind instrument: either change its length, $L$, or change the value of $n$ (we can’t change $a$). Varying $L$ continuously, as in the slide trombone or slide whistle, produces continuous changes in pitch. Alternately, we could change $L$ by making holes in the tube, resulting in discrete changes in pitch. The other way to vary the pitch is to change $n$—that is, to jump between solutions of the wave equation. The discrete set of pitches produced by varying $n$ are the harmonics. Specifically, the pitch with frequency $an/2L$ is called the $n$th harmonic; if $n = 1$ the pitch is the fundamental or first harmonic.

The sequence of ratios of the frequency of the fundamental to the successive harmonics is 1:1, 1:2, 1:3, 1:4, . . . . If the first harmonic is a C, then the next five harmonics are C’, G’, C”, E”, and G”, where each prime denotes the pitch one octave higher. The fourth, fifth, and sixth harmonics form what is called a major chord, one of the primary building blocks of Western music. However, this solution still doesn’t completely explain the willow flute.

Let’s take a closer look at the willow flute player’s right hand. The position of the fingers allows the player to cover or uncover the hole at the end of the flute, thus changing the boundary condition at that end. Solving the wave equation using Calculus, we get a set of solutions for which

$$\text{frequency} = \frac{an}{4L} \quad \text{for} \quad n = 1, 3, 5, 7, \ldots$$

Since the original value of frequency was $an/2L$, closing the end has dropped the fundamental an octave and restricted the harmonics to odd multiples of the fundamental frequency. These two sets of harmonics, along with the ratios of their frequencies to the fundamental of the open pipe, are shown in their approximate positions on a piano keyboard (assuming the fundamental of the open pipe is tuned to a C).