Math for Poets and Drummers

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You've probably heard that "math and music go together". I hear it often; I'm a mathematician and musician who studies the applications of math to music. Precisely *how* math and music go together, though, is somewhat mysterious. There is no doubt that their connection has a long history. The ancient Greeks discovered a correspondence between musical intervals with a pleasing sound and ratios of whole numbers, thus inspiring a persistent association between music and proportions in Western thought. However, the fact that scholars in ancient India made several mathematical breakthroughs while analyzing the rhythms of poetry has been almost completely overlooked in the West.

In English, a poetic rhythm, called a *meter*, is a pattern of stressed and unstressed syllables. English poets use about a dozen different meters. Much poetry, including Shakespeare's plays, is written in *iambic pentameter*—five pairs of alternating unstressed and stressed syllables to a line (da-DUM da-DUM da-DUM da-DUM da-DUM). Alexander Pope's 700+ line iambic pentameter poem *An Essay on Criticism* (1709) is a good example. In this excerpt, he ridicules critics who judge poetry solely on how well it follows strict metrical rules. ("Most" in the first line and "these" in the third line refer to the critics.)

But most by Numbers judge a Poet's Song,
And smooth or rough, with them, is right or wrong;
$\vdots$
These Equal Syllables alone require,
Tho' oft the Ear the open Vowels tire,
While Expletives their feeble Aid do join,
And ten low Words oft creep in one dull Line,

(Even math has meter! The iambic pentameter quadratic $x^2 + 7x + 53$ appears in *Four Riddles* by the mathematician and writer Charles Dodgson, a.k.a. Lewis Carroll.)

While English poets use relatively few meters, there are *hundreds* of meters in Sanskrit, the classical language of India. The struggle to classify such a whopping number of meters inspired several mathematical discoveries: Pascal's triangle, the Fibonacci numbers, and even the rudiments of the binary number system. All of these discoveries were made prior to Western ones.

Pingala (c. 200 BC) was the first of a long line of Indian scholars who studied meter mathematically. In the course of his work on poetry, he discovered "Pascal's" Triangle, which he called the *meruprastāra*, or "the expanding mountain of jewels". (Mt. Meru is a mythical mountain made of gold and precious stones, and *prastāra* is the word for expansion.) Pingala also described an algorithm—that is, a finite set of instructions—that systematically generates all possible metrical patterns. In addition, he explained how to find the precise position of any given pattern on this list. His method is almost identical to how we do binary-to-decimal conversion today. (There is more information on Pingala in an expanded version of this article at `http://www.sju.edu/~rhall`.)

The most significant difference between Sanskrit and English is that Sanskrit meters are based on duration, rather than accent, with a long syllable having twice the length of a short one. This numerical designation led the 12th-century writer Ācārya Hemachandra to consider the following puzzle: How many different meters are there of a given duration?

The solution to this problem is suggested by figure 1, in which 1×1 squares and 1×2 rectangles represent short and long syllables, respectively. The numbers of patterns of each duration form the sequence $1, 2, 3, 5, 8, \ldots$. Hemachandra discovered that each number in the sequence is found by adding the two previous ones. In other words, he found the "Fibonacci" numbers—half a century before Fibonacci! Indian poets and drummers know these numbers as the Hemachandra numbers.
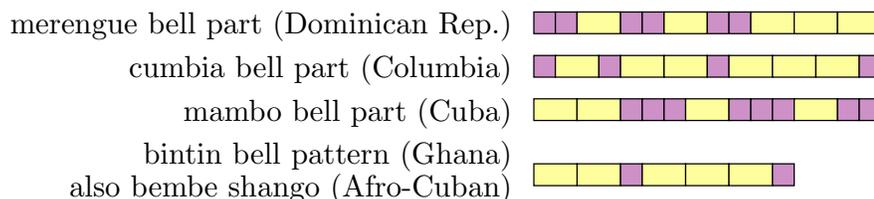
Hemachandra not only commented that the Hemachandra-Fibonacci numbers count meters, but also explained *why* they do so. Partition the collection of patterns of duration

Figure 1: Meters listed by duration. Squares and rectangles represent short and long syllables, respectively.

$n + 1$ into two disjoint sets: patterns of duration n followed by a short syllable and patterns of duration $n - 1$ followed by a long syllable. Let H_n represent the total number of patterns of duration n. The partition shows that $H_{n+1} = H_n + H_{n-1}$. Since $H_1 = 1$ and $H_2 = 2$, we obtain the Hemachandra-Fibonacci sequence.

Hemachandra's problem is equivalent to the "domino-square problem": in how many ways can 1×2 dominoes and 1×1 squares tile a $1 \times n$ rectangle? Figure 2 is a visual proof of the recursion formula.

Figure 2: Visual proof that $H_{n+1} = H_n + H_{n-1}$.

Fibonacci's "discovery" of the sequence that bears his name occurred about fifty years after Hemachandra made his breakthrough. This was probably not a coincidence. Fibonacci, who was educated in North Africa, was quite familiar with Eastern mathematics (his *Liber Abaci* (1202), in which the Fibonacci sequence appears, also introduced the

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Indian positional number system to the West). However, his derivation of the Fibonacci sequence from the sizes of successive generations of rabbits is not found in India.

The poetic meters Hemachandra studied have an analogue in music. *Rhythm patterns* are sequences of drum hits that overlay a steady pulse, or beat. Notes—groups of beats—play the role of syllables in poetry. Drummers hit on the first beat of a note and are silent on the following beats; the length of a note is the number of beats from one hit to the next.

Some types of music, especially dance music, are identified with specific rhythm patterns. Figure 3 shows a few examples. Many of these patterns are composed of one- and two-beat notes. Others consist of two- and three-beat notes. The guajira may be familiar as the rhythm of Leonard Bernstein's "America", from *West Side Story*. To hear more of these patterns, check out SongTrellis (http://www.songtrellis.com/rhythmPage) and Jas's MIDI Hand Drum Rhythm Generator (http://www.khafif.com/rhy/rhygen.html), where you can create your own!

Rhythms of one- and two-beat notes

merengue bell part (Dominican Rep.)

cumbia bell part (Columbia)

mambo bell part (Cuba)

bintin bell pattern (Ghana)
also bembe shango (Afro-Cuban)

Rhythms of two- and three-beat notes

lesnoto (Bulgaria)

bomba (Puerto Rico)
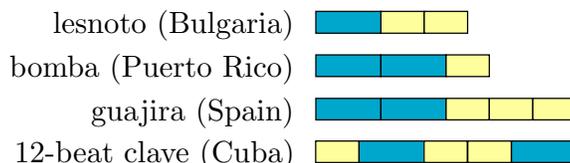
guajira (Spain)

12-beat clave (Cuba)

Figure 3: Dance rhythms

Hemachandra discovered the sequence that counts patterns of one- and two-beat notes. What sequence counts patterns of two- and three-beat notes? Here are the first twelve entries of this sequence, called the Padovan sequence:

length (n)	1	2	3	4	5	6	7	8	9	10	11	12
number of patterns (P_n)	0	1	1	1	2	2	3	4	5	7	9	12

If P_n is the number of n-beat patterns, then $P_{n+1} = P_{n-1} + P_{n-2}$. The proof of this statement is similar to the argument for notes of length one and two. In this case, partition

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the patterns of length $n + 1$ into patterns of length $n - 1$ followed by a two-beat note and patterns of length $n - 2$ followed by a three-beat note.

Though not nearly as famous as the Hemachandra-Fibonacci sequence, the Padovan sequence has some interesting features. Just as the limit of the ratios of successive Fibonacci numbers is the "golden number" $\phi = 1.618033988\ldots$, the limit of the ratios of successive Padovan numbers is the "plastic number" $p = 1.324717957\ldots$. (Oddly enough, the Padovan numbers were first discovered by an architect, but that's a different story.)

To see why the ratios of successive Padovan numbers converge to the plastic number, observe that

$$\frac{P_{n+1}}{P_n} = \frac{P_{n-1}}{P_n} + \frac{P_{n-2}}{P_n} = \frac{P_{n-1}}{P_n} + \frac{P_{n-2}}{P_{n-2} + P_{n-3}} = \frac{P_{n-1}}{P_n} + \frac{P_{n-2}/P_{n-3}}{P_{n-2}/P_{n-3} + 1}.$$

Now suppose the limit $\lim_{n \to \infty} P_{n+1}/P_n$ exists and equals some number x. Take the limit of both sides of the equation as n goes to infinity:

$$x = \frac{1}{x} + \frac{x}{x + 1}.$$

Therefore x (if it exists) must be a solution to the cubic equation $x^3 - x - 1 = 0$. The plastic number p is the only real root of this equation; the other two are complex conjugates. We have now shown that *if* the sequence $\{P_{n+1}/P_n\}_{n=2}^{\infty}$ converges, its limit equals p.

We must now show that the sequence of ratios of Padovan numbers actually *has* a limit. Let q and r represent the complex roots of $x^3 - x - 1 = 0$. With a little work, you can show that the nth Padovan number equals

$$P_n = \frac{p^{n+2}}{3p^2 - 1} + \frac{q^{n+2}}{3q^2 - 1} + \frac{r^{n+2}}{3r^2 - 1}.$$

(This formula has the flavor of the Binet formula for the Fibonacci numbers, and you can prove it in a similar way.) Since $|q| = |r| < 1$, the last two terms approach zero as n goes to infinity, so that $\lim_{n \to \infty} P_{n+1}/P_n$ equals p.

The Padovan sequence has some other beautiful properties—for example, it is related to a spiral of equilateral triangles in the way the Hemachandra-Fibonacci sequence is related to a spiral of squares (figure 4), and it is closely connected to the Perrin sequence (another extremely cool sequence). See Ian Stewart's "Tales of a Neglected Number" (in *Math Hysteria*) for more examples.
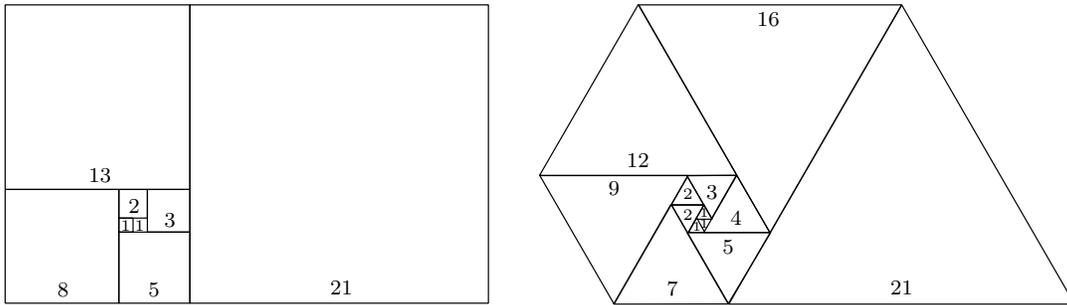
Figure 4: Fibonacci squares and Padovan triangles

Further Reading

Parmanand Singh. The so-called Fibonacci numbers in ancient and medieval India. *Historia Math.*, 12(3):229–244, 1985.

Ian Stewart. *Math Hysteria : Fun and games with mathematics.* Oxford University Press, Oxford, 2004.