Geometrical Models for Modulation in Arabic Music

Abstract

Although Arab music theorists have primarily discussed the static properties of maqāmāt (Arabic melodic modes), modulation between maqāmāt plays a central role in composition, improvisation, and performer training in modern Arab music (Marcus 1989, 1992, 2007). Modulation is achieved by chromatic alteration, change of tonic, or both. The decomposition of each maqām into two or three scalar tetrachords is a primary determinant of its modulatory possibilities. Most modulations involve either a shift in the tetrachord emphasized or a substitution of one tetrachord for another. Although Arab performers and theorists describe maqāmāt as being “close” or “distant” from each other based on the ease of modulation between them, no explicit geometric representations of these relationships exist. In contrast, Western scholars have developed quite a few geometric models representing relationships between Western modes. These include the circle of minor and major keys (Heinichen 1711) and planar networks representing diatonic space (Weber 1821). Neither of these models is satisfactory for Arab music. The chief difficulties are (1) maqāmāt have a lesser degree of symmetry than the modes used in Western music and (2) the sheer number of Arab modes makes their depiction in a planar network difficult. Tymoczko used a voice-leading model to represent relationships between Western scales (Tymoczko 2005). A similar approach succeeds with a central class of Arabic modes. “Voice-leading distance” measures the amount of chromatic alteration required to transform one scale to another (Tymoczko 2006). This notion of distance accords well with studies of Arab musical practice by Marcus (1989, 1992, 2007) and Nettl and Riddle (1973). SplitsTree, an application designed by Huson and Bryant (2006) to compute evolutionary trees or networks, is used to display maqāmāt in networks that represent these distances as accurately as possible. Analysis of Arab modes suggests possibilities for the study of other music that uses the tetrachordal construction, including Turkish, Persian, Indian, ancient and modern Greek, and medieval church music. Moreover, although there are dangers in cross-cultural analysis, this study casts new light on models used in Western music theory.

1. Introduction

The modes of Western music exist within a complex network of relationships. The C major mode is commonly considered to be “close” to the A minor, G major, and C minor modes and “distant” from the C♯ major mode. These examples demonstrate three principles: (1) two modes sharing the same key signature are “close,” (2) two transpositions of the same mode that differ
by a small amount of chromatic alteration are “close,” and (3) two modes sharing the same tonic are “close.” This idea of distance reflects the ease with which a composer can modulate between the two modes. Western music theorists have developed geometrical models to illustrate relationships between modes. For example, Heinichen’s musical circle (Figure 1) represents proximities between the twenty-four Western minor and major modes (Heinichen 1711). Weber located major and minor modes in a toroidal lattice (Weber 1821). No similar models have been developed for Arab music.

![Figure 1. Heinichen’s musical circle (1711). The twelve major modes are adjacent to their relative minor and mediant minor modes.](image)

Arab musicians consider modulatory possibilities to be a central feature of Arab modes, or *maqāmāt* (singular: *maqām*) (Marcus 1992). The concept of proximity between modes based on ease of modulation is well established. Of his own experience learning from Egyptian master musicians, Scott Marcus states,

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1 In this article, “Arab” refers to the Arabic-speaking peoples of Mediterranean countries from Egypt to Syria. Close connections exist to the music of Iraq. Turkish and Persian music are more distantly related.
From these lessons, I learned that each *maqām* is part of a fabric that includes all the *maqāmāt* (or at least a large number of neighboring *maqāmāt*). To know any one *maqām* fully, a student must know all the places to which one can modulate. This stands in marked contrast to Indian music, where the *rāgas* are understood to exist independently (Marcus 1992, p. 175).

My aim in this article is to represent modulatory relationships between *maqāmāt* spatially. In sections 2 and 3, I summarize the construction of *maqāmāt* and the practice of modulation. In section 4, I use the principles of geometrical music theory (Callender, Quinn, and Tymoczko 2008) to generate a three-dimensional graph of the principal four-note subsets of *maqāmāt*. The product of two such graphs arranges a restricted class of *maqāmāt* in a four-dimensional lattice that embeds in a torus. In order to incorporate more *maqāmāt*, I propose, in section 5, a measure of distance that represents, roughly, the amount of chromatic alteration required to move from one *maqām* to another. I use the program SplitsTree to process matrices of distances between *maqāmāt*. This program, which was designed by Huson and Bryant (2006) for evolutionary DNA analysis, creates two-dimensional tree-like networks that represent distances more or less accurately, as measured by least squares fit.

The principal sources for this paper are Scott Marcus’s dissertation (Marcus 1989), article about modulation (Marcus 1992), and book on Egyptian music (Marcus 2007). Marcus’s expertise is in Egyptian music. Other sources include Habib Hassan Touma’s book *The Music of the Arabs* (Touma 1996) and Bruno Nettl and Ronald Riddle’s study of Lebanese musician Jihad Racy’s improvisatory performances (Nettl and Riddle 1973). The excellent web site Maqam World contains discussion of modulation, as well as detailed information about the various *maqāmāt* (Farraj 2007). The chief author of the site is the percussionist and Arab music theorist Johnny Farraj. The Arab violinist, composer, teacher, and theorist Sami Abu Shumays—also a contributor to Maqam World—publishes a podcast of Arabic music lessons (Abu Shumays 2007). Where sources disagree about *maqām* composition, I have considered the sixty-odd *maqāmāt* listed in “Appendix 9: Modal Scales According to Present-Day Theory” on pages 842 to 844 of Marcus’s 1989 dissertation definitive versions. Although there is some danger in relying heavily on one source, it is difficult to find agreement on Arab mode theory, as each author considers slightly different repertoire.
Throughout this paper, uppercase letters refer to notes in the octave from C4 to B4, lowercase letters refer to notes in the octave C5 to B5, and repeated letters raise or lower a note by an octave (so “A” represents A4, “AA” is an octave below A, and “a” and “aa” are one and two octaves above A, respectively). The half-flat symbol “♯” lowers a pitch by a quartertone. I am using the word “note” loosely to mean a “named pitch.” I hesitate to use the more precise word “pitch class” (equivalence class of pitches modulo the octave) because, while Arab music recognizes a relationship between pitches one octave apart, that identification is not as close as in Western music. For example, some Arab scales do not repeat at the octave. With a few exceptions, the names of the notes from GG to G are not related to the names of the notes from G to g. However, outside the range from GG to g, pitches are named in reference to their closest one-octave span. For example, the name for the pitch “aa” means “an octave higher than a,” not “two octaves higher than A.” (See Marcus (1989 p. 99) for a table of “the 49 notes of the modern Arab scale”—49 because Arab musicians use twenty-four-tone equal temperament.)

2. The Construction of Maqāmāt

A maqām is an Arab melodic mode. In contrast to scales, or collections of notes arranged in some ascending order, modes—“weighted scales”—have a tonic and perhaps other important notes. In practice, a maqām is more than a weighted collection of notes—its definition includes a mood, characteristic phrases, starting note, and a typical melodic structure (for example, progressing through its scale by starting a fourth below the tonic, then emphasizing the lower notes of the maqām before developing the upper register). Arab repertoire is primarily monophonic, with consecutive scale steps being more common than other intervals. Maqām theory has a prominence in Arab music theory similar to that of harmony in Western theory.

The theory of maqāmāt is in a state of flux. Modern Arab scale theory—and, in particular, the tetrachordal analysis of maqāmāt discussed in this paper—stems from a twentieth-century revival of medieval Arab tradition (Marcus 1989, pp. 275-279). Prior to the late nineteenth century, maqāmāt were described as collections of melodic phrases rather than sets of notes.

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2 Touma goes farther than this, defining a maqām to be “a technique of improvisation unique to Arabian art music” (Touma 1996, p. 38). This definition is similar to what other authors refer to as “taqāsīm.” Marcus and Farraj give many examples of maqāmāt used in performances that are not improvised.
(ibid., p. 448). While melodic characteristics are still important, the concepts of scale and mode now exist independent of other aspects of maqāmāt. In 1932, the “Committee on the Modes, Rhythms, and Composition” at the Cairo Congress for Arab Music aimed to “establish a scientific theory of Arabic music” (ibid., p. 278). The conference included Western scholars such as Bela Bartok, Paul Hindemith, and Curt Sachs (ibid., p. 31). Although the intervals of Arab music were traditionally defined as frequency ratios in just temperament, the Cairo Congress recommended the use of twenty-four-tone equal temperament (24-tet).³ Western music theory has influenced Arab music in other ways: Marcus writes, “Considering the evidence of new modes being introduced in the late nineteenth and early twentieth centuries … it is possible that the central group of modes has shifted over the last 200 or more years to give more importance to Western diatonic and chromatic modes.” (“Chromatic modes” use the scale of the Western harmonic minor.) The disappearance of most of the maqāmāt that do not repeat at the octave “was undoubtedly influenced by Western scalar theory, a theory which plays a major role in present-day music education” (ibid., p. 536). This article considers modern theory.⁴

Ajnās (singular: jins) are the building blocks of maqāmāt. They span no more than a perfect fifth and typically have four notes, although some have three or five notes (some theorists use the term “tetrachord” to refer to any jins). Figure 2 depicts the nine principal ajnās. They are transposition classes, meaning that they are defined by their sequence of intervals, rather than by absolute pitches. For example, the rāst jins consists of four notes, separated by, in ascending order, a whole tone followed by two intervals of three quartertones. (C, D, E⁵, F) and (G, A, B⁵, c) are both rāst ajnās. Ajnās are determinants of melodies and improvisations, rather than merely subsets of a maqām. Sami Abu Shumays states, “It is very important to think of these small groups of notes … rather than to think of a whole eight-note scale, because many of the melodies

³In practice, only musicians of fixed-pitch instruments such as the accordion use 24-tet; other instruments slightly alter notes in certain maqāmāt.

⁴Modern theorists disagree on how strictly one should define a maqām. Some sources include only one set of notes per named maqām (Marcus’s Appendix 9, “Modal Scales According to Present-Day Theory,” does this). In contrast, Maqam World and Touma list ascending and descending versions of some maqāmāt. Most of these versions are found in Marcus’s Appendix 9, albeit with distinct names. Some musicians use a simpler system, referring to all the maqāmāt whose lower jins is rāst as “maqām rāst” (Marcus 1989, p. 355). I have used only one version of each maqām; however, one can adapt my results by viewing a maqām as a cluster of two or more modes.
are contained within these small units” (Abu Shumays 2007, min. 11:48). Ajnās also play a principal role in modulation, as discussed in section 3.

With a few exceptions, Arab modes are formed from two ajnās. Most maqāmāt sharing the same tonic have similar structure. “C” maqāmāt consist of disjunct four-note ajnās, the lower extending from C to F and the higher extending from G to c (modes with a lower nawā athar pentachord extending from C to G are exceptions). “D” maqāmāt consist of conjunct ajnās, the lower extending from D to G and the higher extending from G to c. “E” maqāmāt consist of conjunct ajnās, the lower extending from E to G or A and the higher from G to c. Less common tonic notes include G, B♭ or B♭, A, and F. Secondary ajnās sometimes overlap the two primary ones. For example, the sīkāh trichord (E, F, G) overlaps the two rāst tetrachords of maqām rāst; this trichord plays a role in modulation between maqām sīkāh and maqām rāst. Figure 3 presents four representative maqāmāt: rāst (a C maqām), bayyātī (a D maqām), huzām (an E maqām), and ṣabā (a D maqām). Most ajnās can appear in either lower or upper positions. The majority of two-jins combinations are represented by at least one maqām. In addition to the tonic, the lowest note in the upper jins, called the ghammāz, plays a special role in performance and modulation. With the exception of ṣabā, most modern maqāmāt (as listed in Marcus 1989) repeat at the octave, although non-repetition at the octave is quite common in historical sources.

Figure 2. Nine principal ajnās.
Although quite a few maqāmāt take their name from their starting pitch (rāst means “C,” jahārkāh means “F,” etc.) and every maqām has a “default” transposition level (e.g. C for maqām rāst), there is some evidence supporting the concept of a maqām as a transposition class. For example, the transposition of rāst to G is called rāst nawā, meaning “rāst on G.” A device described by several late nineteenth- and early twentieth-century Arab music theorists provides further evidence. This device—a so-called “musical compass”—was formed of two wheels, each with the 48 notes of the fundamental two-octave range written around its circumference. To find the notes of a transposed maqām, one located the original maqām on one wheel, aligned the original and new tonic notes, and read off the transposed pitches on the second wheel (Marcus 1989, p. 108-109). This device did not catch on in the Arab world. It is particularly interesting because of the lack of a circular model in Arab modulation theory.

In practice, transposition can occur to accommodate a singer or simply for variety. In this case, the collection of maqāmāt that are available to the performer is transposed by the same interval (Marcus 1989 p. 733). In addition, as described in the next section of this article, modulation sometimes requires transposition. Although the adoption of 24-tet makes transposition of
maqāmāt to any starting pitch theoretically possible, transpositions most commonly occur by fifths or fourths, corresponding to the tunings of common stringed instruments.

3. Modulation

Modulation is the process of changing mode during the course of a piece. It is central to the performance of taqāsīm—improvised presentations of a maqām in free rhythm—and appears in more structured vocal and instrumental music. For example, fifteen of the sixteen taqāsīm by Jihad Racy that Nettl and Riddle analyzed contain modulation; the one lacking modulation is by far the shortest (Nettl and Riddle 1973). As in Western music, modulation in Arab music can occur in a number of ways. It sometimes involves changing the location of the tonic (for example, moving from C to E♭, as in the modulation from rāst to sīkāh). Chromatic alteration may also occur. Although it is possible for both ajnās to change, a modulation in which only one jins changes—and, in particular, modulation in which the upper jins changes—is considered the least disruptive (Farraj 2007). Arab performers use metaphors of distance to describe modulation. The musicians that Marcus interviewed emphasized, “The modes stand in different levels of proximity to one another. The relationship between any two modes is usually expressed in terms of the adjectives ‘close,’ ‘closer,’ and ‘distant’” (Marcus 1992 p. 183).

Musicians often perform a sequence of modulations in order to reach certain maqāmāt. This practice further supports the idea that Arab modes exist in a “fabric” rather than in isolation. In order to modulate to a “distant” maqām, one normally interpolates a chain of maqāmāt that are relatively “close” to each other. Rare maqāmāt come at the end of a sequence of modulations (Nettl and Riddle p. 19-20). One musician Marcus interviewed stated, “Modulation between šabā and hijāz is not possible except after passing through [an unnamed mode whose characteristics are between those of šabā and hijāz]” (Marcus 1992, p. 182). Another stated, “When modulating from shūrī to rāst nawa [rāst on G] it is desirable to insert bayyātī as an
intermediary mode” (ibid., p. 182). Marcus writes, “An example of an unusual modulation to a distant mode would be a direct move from rāst to ʂabā nawā. More commonly, one would first modulate from rāst to bayyātī nawā, and then to ʂabā nawā” (ibid., p. 181).

The most commonly cited “rule” of modulation is that one must return to the starting maqām at the end of a piece. Marcus make four additional observations about which modulations do and do not typically occur (Marcus 1992, pp. 186-188). These rules correspond to musicians’ comments on which maqāmāt are close or distant from each other—that is, maqāmāt are “close” if one can modulate between them without any intermediate steps.

1. Modulation between maqāmāt with the same tonic is common.

2. Modulation between maqāmāt whose tonic is C and those whose tonic is D only occurs when one maqām to is transposed to G (nawā). For example, one would modulate either from rāst to bayyātī nawā or from bayyātī to rāst nawā but not from rāst to bayyātī (Figure 4).

3. The E♭ (or E♭) and B♭ (or B♭) maqāmāt modulate with the C and D maqāmāt, respectively. An E♭ maqām can be transposed to B♭ in order to modulate to or from a D maqām, and a B♭ maqām can be transposed to E♭ in order to modulate to or from a C maqām.

4. The F maqāmāt can modulate with B♭, C, or D maqāmāt. Modulations between F maqāmāt and the C modes ḥijāz kār or ḥijāz kār kurd are particularly common.

From rāst, the most common modulations are to other C maqāmāt (e.g. sūznāk and nahāwand), E♭ maqāmāt such as sīkāh, and D maqāmāt transposed to G such as bayyātī nawā (ibid., p. 180).

These rules are evidence for the prominent role that ajnās play in modulation. Abu Shumays describes modulation as a substitution of one jins for another:

The way that maqāms move back and forth is in these [ajnās]. … Now, what happens in Arabic music, and in a maqām, is that you start in one jins, and then you move to another area. (Abu Shumays 2007, min. 11:48)

In his discussion of modulation, Farraj states that the most common form of modulation involves either substituting one upper jins for another or one lower jins for another, using the ghammāz as the “pivot note.” Other modulations arise from development of secondary ajnās within a maqām (Farraj 2007, www.maqamworld.com/modulation.html). He describes how one can modulate
from bayyātī on D to 'ajam 'ushayrān on B♭ by emphasizing the first three notes of the 'ajam tetrachord (B♭, C, D) within bayyātī and then developing this embedded jins into a full maqām. We can extrapolate Marcus’s rules from this description. Since B♭ has the relation to the upper jins of the D modes that E♭ has to the lower jins of the C modes, one can use the same process to modulate from a C to an E♭ or E♭ maqām. Moreover, modulation from rāst to bayyātī nawā can occur in three stages: (1) develop the upper jins of rāst, (2) substitute the bayyātī jins on G for the rāst jins on G, and (3) present the entire maqām bayyātī nawā.  

Farraj does not separate maqāmāt into C and D families, but rather into “those whose dominant note is the fifth” and “those whose dominant note is the fourth” (ibid.). (One is tempted to make the connection to “authentic” and “plagal” church modes.) He states that modulation between these two families is uncommon, although he does not suggest transposition to G as a solution.

4. Geometrical music theory and scale lattices

Although music theorists have used geometry to model musical relationships for centuries, so-called “geometrical music theory” is a relatively recent theory developed by Clifton Callender, Ian Quinn, and Dmitri Tymoczko (Callender 2004, Tymoczko 2006, Callender, Quinn, and Tymoczko 2008). Geometrical music theory recognizes that any musical object that can be represented by an n-tuple of pitches corresponds to a point in some n-dimensional Euclidean space. Common equivalence relations, such as octave equivalence, define quotient maps on Euclidean space producing a family of singular, non-Euclidean, quotient spaces—orbifolds—that

\[ \text{Figure 5. The voice-leading distance between the keys of F and B is a function of the multiset of changes \{1, 1, 0, 1, 1, 1, 1\}.} \]
subsume many geometrical models previously proposed in the music theory literature. The basic definitions of geometrical music theory are as follows: *Pitch* is frequency measured on a logarithmic scale, with twelve units of pitch (semitones) to the octave. Pitches lie on a continuum; integer pitches form twelve-tone equal temperament (12-tet). A *pitch class* is an equivalence class of pitches, where two pitches lie in the same pitch class if they are separated by a whole number of octaves. Individual pitch classes lie on a circle. A *chord* is an unordered collection—a *multiset*—of pitch classes, while a *scale* is a collection of pitch classes arranged in some ascending order on the pitch class circle. A *mode* is a scale with a distinguished tonic. The set of transpositions of a given chord or scale form its *transposition class* or *scale class*, respectively. For example, if we assign “0” to pitch class “C,” the 7-tuples (0, 2, 3.5, 5, 7, 9, 10.5) and (5, 7, 9, 10.5, 0, 2, 3.5) represent the same scale: the scale of *maqām rāst*, also called the “fundamental scale” of Arab music. Addition modulo twelve corresponds to transposition, so (2, 4, 5.5, 7, 9, 11, 0.5) belongs to the scale class of the fundamental scale. Each equivalence class embeds in the quotient of Euclidean space defined by the corresponding “OPTI” equivalence relations—Octave equivalence, Permutation, Transposition, and Inversion (see Callender, Quinn, and Tymoczko 2008 for detailed descriptions of these quotient spaces).

Although we can think of individual *maqāmāt* as sets with a distinguished tonic, in order to model modulation we must consider how the notes of one *maqām* are altered to produce another *maqām*. A *voice leading* is a relation between two multisets of pitches or pitch classes, where each pitch in the “source” is paired with some pitch in the “target” and vice versa. I consider only bijective voice leadings in this article. The notation \((a_1, \ldots, a_n) \rightarrow (b_1, \ldots, b_n)\), where the \(a_i\)'s and \(b_i\)'s are pitch classes, means that pitch class \(a_1\) is paired with \(b_1\), \(a_2\) is paired with \(b_2\), and so on. Two voice leadings are equivalent if they pair the same pitches. For example, \((C, E, G) \rightarrow (C, F, A)\) and \((E, C, G) \rightarrow (F, C, A)\) are equivalent. Tymoczko (2005) proposed that key changes are, in fact, voice leadings, because the flat and sharp symbols in a key signature indicate a directional mapping of one pitch class to another. For example, modulation from the key of \(F\) to the key of \(B\) corresponds to the voice leading \((F, G, A, B\flat, C, D, E) \rightarrow (F\# G\#, A\#, B\flat, C\#, D\#, E)\), which fixes E and moves F up to F#. G up to G#, and so on. Figure 5 depicts this voice leading. However, changing keys from \(F\) to \(C\##\) involves a different voice leading: \((F, G, A, B\flat, C, D, E) \rightarrow (F\flat G\flat, A\flat, B\flat, C\flat, D\flat, E\flat)\). In this case, \(B\flat\) is the only fixed pitch class and
all others move down a semitone. This distinction is also relevant in Arab music. For example, when substituting a ḥijāz jins (D, E♭, F♯, G) for a ṣabā jins (D, E♭, F, G♭), one pairs the F♯ with F rather than G♭.

Since the scales commonly used in Western and Arab music lie in a seven-dimensional space, successfully representing them in three dimensions is possible only in cases with a high degree of symmetry. The Pressing scales are the only seven-note scales in 12-tet that satisfy the properties that (1) they contain no consecutive semitones and (2) their scalar thirds measure three or four semitones. (These two properties are related—Tymoczko (2004) showed that “maximal sets” that satisfy (1) correspond to “minimal sets” that satisfy (2).) Pressing scales lie in the elegant lattice of Figure 6 precisely because they are generated by a particular algorithm.
I will comment on the relationship between Pressing scales and Arab scales in the following pages.

*A lattice model for ajnās and maqāmāt*

Due to the great variety of scales used, *maqāmāt* cannot be easily depicted in a lattice; however, *ajnās* are manageable. Replacing one *jins* with another always preserves the lowest note of the *jins*. Therefore, a substitution involving four-note *ajnās* corresponds to some voice leading of the type \((x, y_1, z_1, w_1) \rightarrow (x, y_2, z_2, w_2)\), where the coordinates of each *jins* are ordered from its

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6 Although this graph appears three-dimensional, representing the lattice accurately requires seven dimensions. Note, for example, that the D diatonic scale does not lie on a line between the G diatonic scale and the B harmonic minor scale in seven-dimensional space. Moreover, the lattice continues in both directions in order to embrace different keys.
lowest note. The lattice shown in Figure 7 represents ajnās with the same lowest note \(t\). They are contained in the three-dimensional subspace \(x = t\). The four-note sets that span a perfect fourth \(\{t + (0, y, z, 5)\}\) lie in a two-dimensional subspace. Note that the even division of a perfect fourth \(t + (0, 5/3, 10/3, 5)\) lies closest to rāst and bayyātī. Since no jins lies too far from an even division of the fourth, the overlapping jins structure ensures that all the seven-note scales of maqāmāt that are formed of two tetrachords divide the octave more or less evenly. Note also that all the ajnās satisfy Tymoczko’s “no consecutive semitones” rule; moreover, their scalar thirds are between three and four semitones wide.

We can locate the scales that are the union of two four-note ajnās in the product of the lattice of Figure 7 with itself. For the moment, let us disregard the šabā and nawā athar ajnās. The remaining ajnās lie on a square, and the product of this square with itself is topologically a torus (note that this particular torus cannot be represented in three dimensions without distorting distances). We construct two tori (Figure 8), the left one representing the C maqāmāt formed of disjunct ajnās, and the right one representing the D maqāmāt formed of conjunct ajnās. These tori have been flattened into planar graphs—one should imagine that their parallel edges are glued together. Suppose we represent every mode by a point in seven-dimensional space \((x_1, x_2, \ldots, x_7)\). All the C maqāmāt in this lattice contain the notes C, F, and G. Therefore, the lattice containing these maqāmāt lies in the four-dimensional subspace defined by \(x_1 = 0, x_4 = 5,\) and \(x_5 = 7\). Likewise, the lattice containing the D maqāmāt lies in the four-dimensional subspace \(x_1 = 2, x_4 = 7,\) and \(x_5 = 12\). We can add a few more maqāmāt by attaching the nawā athar tetrachord, which appears as a lower jins in C maqāmāt and an upper jins in D maqāmāt. I will refer to the modes of Figure 8 as the “regular” C and D maqāmāt.

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7 Translating all the ajnās to the same starting note is somewhat misleading, since one would not modulate from šabā on C to nawā athar on C—šabā occurs as the lower tetrachord in the D family, and nawā athar appears as the lower tetrachord in the C family of maqāmāt.

8 In fact, rāst and bayyātī are “maximally even” divisions of a perfect fourth in 24-tet, in the sense of Clough and Douthett (1991).

9 The vertices of the product lattice are the ordered pairs of vertices \((v_1, v_2)\), where each \(v_i\) corresponds to a jins.
Figure 8. The regular C and D maqāmāt located on toroidal lattices.
Modulation

The practice of modulation requires one to move between the C and D modes by transposing one collection (say, the D modes) to G so that its upper tetrachords align with the lower tetrachords of the other collection. Figure 9 represents the superimposition of the two lattices, with the D lattice rotated by 90°. The regular C modes appear horizontally and the regular D modes appear vertically. If we transpose the D modes to G as in Figure 10, maqāmāt that correspond to the same point consist of the same pitch classes, albeit with different tonics. For example, the pitches of nahāwand kabīr are (C, D, E♭, F, G, A, B♭, c) and the pitches of ‘ushāq masrī nawā I are (G, A, B♭, c, c, d, e♭, f)—in other words, duplication of pitch classes occurs where the two conjunct tetrachords overlap. Likewise, nahāwand kabīr nawā and ‘ushāq masrī I share the same pitch classes. I have added a few more maqāmāt to Figure 10 by locating modes whose tonic is E♯ or E♭ (or transposed B♯ or B♭ modes) at the point representing the modes whose pitch classes they share. About two-thirds of the maqāmāt listed in Marcus’s Appendix 9 either appear in Figure 10 in their original form or appear transposed up a fourth. I will refer to these as “regular” maqāmāt.
The toroidal lattice of Figure 10 represents relationships between modes spatially. C or D *maqāmāt* that lie on the same vertical or horizontal line share either an upper or a lower *jins*. For example, all the *maqāmāt* with a disjunct lower *rāst* tetrachord on C lie on the same horizontal line. All the D *maqāmāt* with a conjunct lower *bayyāṭī* tetrachord lie on a vertical line; likewise, all the C *maqāmāt* with a disjunct upper *nahāwand* tetrachord lie on a vertical line. Therefore, modulations along horizontal or vertical lines are common. Shifting emphasis on the *ajnās* within a mode creates modulation without chromatic alteration. It corresponds to tilting the lattice so a different set of modes appears horizontally. This graph indicates which chromatic alterations are required to modulate from one *maqām* to another. For example, modulating from *rāst* to *bayyāṭī* *nawā* involves three quartetone alterations (recall that *jins* *nahāwand* is a corner of the square in our *jins* graph (Figure 7)—therefore, one makes two separate quartetone alterations between the *rāst* and *bayyāṭī* *ajnās*, rather than one semitone alteration). Figure 4 confirms this reading of the graph. *Maqāmāt* that are intermediate between *rāst* and *bayyāṭī* *nawā* include *sūzdilār*, *nīrz*, *nahāwand* *kabīr*, and both versions of *ʿushāq* *masrī* *nawā*. The graph shows that no two regular *maqāmāt* are farther from each other than *nahāwand* *kabīr* and *ḥijāz* *kār*; four chromatic alterations of one semitone each are required to move between these (several other pairs, such as *zingarān* and *ḥijāz*, have this relationship). Moreover, no mode is too far isolated from the others. The only regular *maqāmāt* that are more than a semitone from their nearest neighbors are *ḥijāz* *kār* and *ḥijāzayn*.

How well does this lattice model modulatory practice? One sequence of modulations mentioned in section 3 is *shūrī*→*bayyāṭī*→*rāst* *nawā*. According to Marcus, one inserts *bayyāṭī* because *shūrī* and *rāst* *nawā* are considered “distant.” Six modes lie in a direct path on the lattice from *shūrī* to *rāst* *nawā*, including *husaynī*, *nīrz* *nawā*, and *sūzdilār* *nawā* (recall that the lattice is essentially the same when we transpose the C modes rather than the D modes). However, none of these is particularly common, while *bayyāṭī* is one of the most common modes. *Bayyāṭī* is closer to both *shūrī* and *rāst* *nawā* than they are to each other. Although there may be other reasons why musicians use *bayyāṭī* as an intermediate mode, this practice avoids large amounts of chromatic alteration in one modulation.
Figure 10. The 38 regular C, D, and E or B maqāmāt that are listed in Marcus 1989, pages 842-844. Maqāmāt written in horizontal text are formed of disjunct tetrachords, with the lower arranged in rows and the upper arranged in columns. These maqāmāt have C as their tonic (the Western C major mode contains two ‘ajam ajnās). Vertical text indicates maqāmāt with conjunct ajnās. These have their tonics on G. Names written at a 45-degree angle represent E or B maqāmāt or B maqāmāt transposed to E. With the exception of hijāz kār/hijāzayn, no regular maqām lies more than one semitone from all the other regular modes.
The common scale classes of Arab music

The lattice of regular maqāmāt has a good deal of symmetry—in fact, the scales corresponding to its vertices belong to eleven scale classes, four of which are Pressing scales. As in Tymoczko’s Pressing scale lattice (Figure 6 of this paper), each square of the lattice represents a cycle of semitone alterations. Figure 11 reveals the relationships between the scales of the regular maqāmāt and the Pressing scales, which are the corner points of the shaded squares. Of particular interest are the six “principal Arab scale classes” that are each represented by six or more maqāmāt: the scale classes of rāst, bayyātī, hijāz kār, and sūznāk (which are not Pressing scales) and the Western diatonic and harmonic minor scales (which are Pressing scales). Of these, only hijāz kār is neither a Pressing scale nor an interpolation between two Pressing scales.

Figure 11. Scales of Arab music on a toroidal lattice. Not every maqām is represented (for example, the athar kurd scale is omitted because it contains a unique jins). Each arrow represents a maqām. Shaded squares are faces in Tymoczko’s lattice of Pressing scales (Figure 6).
The scale classes used in regular maqāmāt are not too different from those of maqāmāt that are not regular. Figure 12 shows the scale classes of all the modes in Marcus’s Appendix 9 on pitch class circles. (Modes with asterisks belong to scale classes that are inversions of the ones depicted.) Non-octave repeating modes appear at the bottom of the figure; each of the sabā variants is a subset of another non-repeating mode. The following is a summary of the scalar interval content of the maqāmāt represented in Figure 12:

**Scale steps:** With the exception of the quartetone D♯-E♭ interval in sāzkār, scale steps measure between one and three semitones.

**Thirds:** With two exceptions, no scalar third measures more than four or less than two semitones. This ensures that an augmented second is both followed and preceded by a semitone. Athar kurd, which has a five-semitone third, and sāzkār, which has a nine-quartertone third, are exceptions.

**Fourths and fifths:** Scalar fourths vary from four to six semitones and fifths vary from six to eight semitones. Every scale contains at least two consecutive scalar fifths that are perfect fifths.

**Consecutive semitones:** Consecutive semitones are adjacent to at least one augmented second.

**Microtonal intervals:** Non-12-tet scale steps belong to isolated pairs of consecutive microtonal steps.

Any scale that follows these “rules” will be fairly even.

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10 Consecutive scale steps determine scalar intervals. A *scalar third* is the union of two consecutive scale steps; a *scalar fourth* is the union of three consecutive steps; and so on. I will refer to intervals in the non-octave-repeating modes as “scale steps” even though these modes are not technically scales.

11 In the case of octave-repeating modes, this statement is equivalent to “every scale contains at least two perfect fourths.” However, the three sabā-related modes do not contain consecutive fourths.

12 Consecutive semitones only occur on the boundary between two conjunct ajnās. Their function is to tonicize the tonic or the ghammāz (Marcus 1989, p. 617). Tymoczko (2004) notes the implications of consecutive semitones in triadic music: if a “scalar triad” is a chord containing two scalar thirds, then scales containing consecutive semitones contain triads with a third measuring a whole tone. However, Arab music places no emphasis on triads.
Figure 12. Maqāmāt belong to a large variety of scale classes. I have ordered these by the size of their minimal voice leading to a perfectly even division of the octave, as measured by the $L^2$ norm (i.e. Euclidean distance). The maximally even scale class is provided for comparison. Relative “unevenness” explains, at least in part, why maqāmāt such as athar kurd, sāzkār, and the various maqāmāt belonging to the ḥijāz kār scale class are outliers in Figure 16. An asterisk (*) indicates modes whose scale class is the inversion of the one depicted. The six sabā-related modes at the bottom of the figure are not technically scale classes because they do not repeat at the octave.
5. Distance geometry and SplitsTree analysis

There are several ways to measure distances between scales. In this paper, I consider measurements based on the voice-leading model of modulation. Tymoczko (2006) proposed that any measure of voice leadings should depend on the multiset of absolute distances traveled by each voice, which is called the displacement multiset of the voice leading. The distance between keys depends on the magnitudes of the chromatic alterations required to transform one to another, preserving letter names. For example, in modulation from the key of F to the key of B, the displacement multiset is \{0, 1, 1, 1, 1, 1\}. This measure of distance does not differentiate between modes that share the same scale, such as C major and A minor (or rāst on C and sikāh on E♭). Tymoczko proposed that all “reasonable” voice leading measures should respect two basic principles: (1) crossed voice leadings are less efficient—and therefore have a greater “measure”—than their uncrossed alternatives and (2) increasing the amount of change in any one voice increases overall change in the voice leading. “Reasonable” voice-leading measures include \(L^p\) norms for \(1 \leq p \leq \infty\). The \(L^1\) norm, which simply sums the members of the displacement multiset, is the most commonly used of these. One attractive feature of this norm is that it measures an obvious quantity: the total amount of chromatic alteration required to transform one scale to another. Using this model, the \(L^1\) distance from the key of F to the key of B is 6 semitones. The \(L^\infty\) norm measures only the largest distance traveled by some voice, so that the distance from F to B is 1 semitone. This measurement is normally too crude for computing distances between scales, as modulations typically involve only a few different sizes of displacements. For example, the key of C lies at \(L^\infty\) distance 1 from all the keys with fewer than eight flats or sharps. The \(L^2\) distance is the square root of the sum of the squared elements of the displacement multiset, so that the distance from F to B is \(\sqrt{6}\). In this project, I have found the \(L^1\) distance to produce the most elegant visualizations. Moreover, since modulation in scalar music occurs “one step at a time,” rather than all at once, this measure makes good sense musically.

The chief problem with the lattice of Figure 10 is that it only accommodates the “regular” modes, thus omitting quite a few common modes such as \(\text{sabā}\). The program SplitsTree, written by Huson and Bryant (2006), constructs graphs from matrices of distances between “taxa”

\(^{13}\)The \(L^p\) norm of a multiset \(\{x_1, \ldots, x_n\}\) equals \((|x_1|^p + \cdots + |x_n|^p)^{1/p}\) for \(0 < p < \infty\). The \(L^\infty\) norm is defined to be the largest element of \(\{|x_1|, \ldots, |x_n|\}\).
(biological organisms). A splits graph is a “tree-like” connected planar network representing distances between taxa, which appear as “leaves,” or nodes of degree one or two. Although evolutionary relationships are commonly described as trees (graphs with no loops), network analysis is useful where we lack sufficient evidence or have conflicting information. The distance between any two taxa is approximated by the sum of the lengths of the edges in the shortest path connecting them. The “least squares fit” of the splits graph equals the sum of the approximated distances divided by the sum of the original distances (Huson and Bryant 2006, p. 69). Therefore, although the SplitsTree algorithm attempts to represent distances accurately, networks with a high least squares fit may not approximate every distance well. Toussaint (2003) was the first to use SplitsTree in musical analysis—specifically, in his investigation of African rhythm timelines.

Figure 13 takes the most naïve approach to modeling the relationships between maqāmāt. The $L^1$ distances between seventeen common maqāmāt (as identified in Marcus 1992, p. 185) are entered into SplitsTree. Distances were averaged over two octaves in order to include ṣabā. The resulting network represents the amount of chromatic alteration required to modulate from one maqām to another. This network does a poor job representing modulatory practice. Rāst on C is closer to bayyātī on D in the graph than it is to any other maqām other than sīkāh, which shares
its scale. However, according to all accounts, they are considered “distant” because modulation between the two is rare.

Voice leading distance does not explain the avoidance of modulation between C maqāmāt and D maqāmāt. If we merely consider rāst and bayyātī as scales, modulating from rāst on C to bayyātī on D (an atypical modulation) requires one quartetone alteration: the B♭ in rāst becomes a B♭ in bayyātī (see Figure 3). In contrast, the common modulation from rāst to bayyātī transposed to G requires three quartetone alterations (see Figure 4). In this sense, bayyātī on D is closer to rāst than bayyātī on G is to rāst. However, if we consider a modulation to be the substitution of one jins for another—a voice leading between ajnās rather than scales—rāst on C and bayyātī on D are far apart. The upper jins voice leading (7, 9, 10.5, 12)→(7, 9, 10, 12) is still small, but the lower jins voice leading (0, 2, 3.5, 5)→(2, 3.5, 5, 7) has the displacement multiset \{2, 1.5, 1.5, 2\}.

Figure 14. Here, the common regular C and D maqāmāt are considered as multisets with repeated notes where tetrachords overlap (Fit = 98.91%). The C maqāmāt cluster with transpositions of the D maqāmāt to G and vice versa.
Many musicians describe modulation as substitution of one *jins* for another. In Figure 14, each *maqām* is represented by the multiset that is the disjoint union of its two *jins*. For example, the multiset \{0, 2, 3.5, 5, 7, 9, 10.5, 0\} represents *rāst* on C, while \{2, 3.5, 5, 7, 7, 9, 10, 0\} represents *bayyātī* on D. Modulation from *rāst* to *bayyātī* is represented by the voice leading \((0, 2, 3.5, 5, 7, 9, 10.5, 0)\rightarrow(2, 3.5, 5, 7, 7, 9, 10, 0)\), which has displacement multiset \{2, 1.5, 1.5, 2, 0, 0, 0.5, 0\}. In contrast, the modulation from *rāst* to *bayyātī* *nawā* corresponds to the voice leading \((0, 2, 3.5, 5, 7, 9, 10.5, 0)\rightarrow(0, 2, 3, 5, 7, 8.5, 10, 0)\), which has the displacement multiset \{0, 0.5, 0, 0, 0.5, 0.5, 0\}. Figure 14 shows that the \(L^1\) norm of these “multiset” voice leadings successfully clusters the C *maqāmāt* with the transposed D *maqāmāt* and the D *maqāmāt* with the transposed C *maqāmāt*.

Since the C and D *maqāmāt* offer similar modulatory possibilities, I consider modulation among the “C family” of *maqāmāt*, meaning those to which one can modulate from *maqām rāst*. This category includes the *maqāmāt* whose tonics are C, E\#, E♭, or F and transpositions up a fourth of the *maqāmāt* whose tonics are D, B\#, or B♭. Marcus (1992) also mentions *ṣabā* on A as a
possible modulation from rāst. Figure 16 shows a network consisting of fifty-five maqāmāt from Marcus 1989, Appendix 9. Distances were averaged over two octaves in order to incorporate ṣabā and the other non-repeating scales. Although the graph is quite complex, a few features emerge: rāst, bayyātī nawā, and nahāwand—are three of the most common maqāmāt—are near the center of the network, where the greatest number of maqāmāt may be reached by minimal effort. These maqāmāt are close to even divisions of the octave. Figure 15 shows only the seventeen common maqāmāt, transposed if necessary to modulate to rāst.

The geometry of maqāmāt creates problems for SplitsTree, which depicts taxa in a planar network. However, SplitsTree does a good job “locally” of isolating groups that are mutually close to each other and relatively equidistant from others. For example, it groups the “A” ṣabā modes with ḥijāz kār. (Ṣabā on A has a lower ṣabā tetrachord on AA followed by the ḥijāz kār scale. Moreover, Figures 15 and 16 show that bayyātī nawā is the closest common mode to ṣabā nawā. This corresponds to the observation that bayyātī nawā is inserted between rāst and ṣabā nawā.

One of Toussaint’s motivations in his phylogenetic analysis of African rhythms was to uncover an “ancestral” rhythm—one that generates the other rhythms with a small amount of “mutation.” In the study of modes, mutation corresponds to chromatic alteration. Is there an Arab mode that occupies a similar position with respect to the other maqāmāt? Study of Figures 15 and 16 show that quite a few modes are central to the network of maqāmāt. These include rāst, sīkāh, bayyātī nawā, nahāwand, and other common modes. The average distance to the other modes is minimized by two scales: the scale of bayyātī nawā and ‘ajam murassa‘ transposed to E♭ and the scale of nīrūz, ‘irāq, and ḥusaynī nawā (indicated by the large dot in Figure 16). Their average distance to the 55 maqāmāt in the figure is 1.62 semitones. The scale of nīrūz, ‘irāq, and ḥusaynī nawā, (C, D, E♭, F, G, A♭, B, c), combines disjunct rāst and bayyātī tetrachords and has the smallest minimal distance (three semitones) to these 55 modes. In other words, any of these 55 modes can be generated by a “mutation” of nīrūz, ‘irāq, and ḥusaynī nawā of no more than three semitones total. Bayyātī is the “most central” of the common maqāmāt; it lies no more than 3.5 semitones from any other mode. Note that bayyātī is twice mentioned as an intermediate mode in Marcus 1992.
There is a cluster of more or less even maqāmāt, including rāst and bayyātī nawā. Others, such as athar kurd, are outliers. However, no maqām is more one semitone’s distance from all of the others. The appearance of outliers is related to the fact that maqāmāt belong to a large variety of set classes, some of which are quite uneven (Figure 12). The least-squares fit is 95.95%. Note that some distances are poorly represented. For example, ḥijāz kār should be closer to shūrī and sūznāk than it is to ḥijāz kār kurd.
One disadvantage of voice-leading distance is that it does not separate maqāmāt with the same notes but different tonics (for example, rāst and sīkāh). One can add a “penalty” for change of tonic, so that the distance from one maqām to another is the amount of chromatic alteration plus some value $p$ if the two maqāmāt have different tonics. Figure 15 represents $p = 0$ (there is no penalty for changing tonic). Setting $p = 1$ produces only a slight change in the graph (Figure 17, top right)—just enough to separate rāst from sīkāh, but not enough to move rāst far from bayyātī nawā. Setting $p = 3$ produces a network with some clustering by tonic (bottom left in the figure). Clusters of maqāmāt sharing the same tonic are clearly apparent when $p = 5$ (bottom right). All of these networks model the preference for inserting bayyātī nawā before a modulation to ṣabā nawā.

6. Conclusion

Figure 10 (a toroidal lattice) and Figure 16 (a SplitsTree network) model the “fabric” of maqāmāt based on a mathematical interpretation of modulations as voice leadings between ajnās. Although the SplitsTree network model has some attractive features, geometrical music theory delivers a more honest—though more visually difficult—portrayal of the complicated voice-leading relationships between Arab modes. Arab modes do not exist in a “tree-like” planar structure but rather a seven-dimensional “web” in which there are multiple paths between maqāmāt. By clearly showing the decomposition of regular maqāmāt into tetrachords, the toroidal lattice reveals a structure of modes related by similar tetrachord composition. Goals for future work include using statistical analyses of performances to test the models and extending the techniques of this article to other music that uses the tetrachordal construction, including Turkish, Persian, Indian, Ancient and Modern Greek, and medieval church music. Moreover, this investigation has raised questions that I hope to answer about using SplitsTree to model musical relationships.

Mathematics—in particular, “geometrical thinking” and a preference for symmetry—has profoundly influenced the course of Western music theory. It is already clear that exposure to

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14 Lerdahl takes this approach; however, he sets the penalty equal to the number of fifths by which the tonic changes. Therefore, the penalty for a one-semitone alteration is equal to the penalty for a change of tonic by a fifth. This attitude is unnecessarily simplistic—I prefer to use parameters to “tweak” the models rather than assume these two alterations have the same weight.
Western theory has changed the course of Arab music in the modern era. The shift to equal temperament, the concept of scale, and the emergence of modes using Western scales all date to the middle-late twentieth century. The Western circular model of modes has profound implications in Western music. In theory, one can modulate between any two Western modes. In Arab music, however, the tetrachordal structure of the starting mode determines the available targets of modulation. Since adopting a Westernized theory of modulation would require relaxation—or even abandonment—of the distinctive tetrachordal decomposition of Arab modes, it seems unlikely that a circular model would become popular.
Figure 17. Networks showing the effect of adding a penalty \( (p) \) for changing tonic. From the top left, \( p = 0.5, p = 1, p = 1.5, \) and \( p = 2 \). The least-squares fit is greater than 95% for all of these networks.
Bibliography


