A Course in Multicultural Mathematics

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Abstract

The course described in this paper, *Multicultural Mathematics*, aims to strengthen and expand students' understanding of fundamental mathematics—number systems, arithmetic, geometry, elementary number theory, and mathematical reasoning—through study of the mathematics of world cultures. In addition, the course is designed to explore the connections between mathematics and the arts, to engage students' imagination and creativity, and to increase the diversity of offerings in the mathematics classroom. This article details a course in multicultural mathematics for liberal arts and education majors I have been teaching for several years. The first three sections describe the rationale, structure, and main topics of the course. Sample projects and questions for class work and discussion are provided in the final two sections. An extensive source list is included.

Keywords

Multicultural mathematics, ethnomathematics, liberal arts, number systems, mathematics of art, mathematics of music.

1 Introduction

The goals of the course *Multicultural Mathematics* are to study the mathematics of selected world cultures and to examine mathematical structures inherent in folk arts and music. *Multicultural Mathematics* grew out of my desire to share some of my own passions—mathematics, languages, folk music, and folk culture—with my students, and to give them an appreciation of the richness of mathematical culture. The first three sections of this paper describe the rationale and structure of the course. In order to give the reader an idea of the actual "flavor" of the course, the final two sections provide sample projects and questions for exploration.

The study of multicultural mathematics reflects two challenges to Eurocentrism in mathematics. One, typified by Joseph's *Crest of the Peacock* [21], seeks to recognize the contributions of non-Western peoples to the history of mathematics. The other, represented by the work of Ascher and Ascher [2, 5, 3], explores mathematical thinking—"ideas . . . involving number, logic, spatial configuration, and, more significant, the combination or organization of these into systems and structures," [4, p. 25]—found outside what we traditionally consider "mathematics."[1] I will refer to this type of mathematical thinking as *ethnomathematics*, as opposed to "academic mathematics." For more on ethnomathematics, see [29].

Several authors have addressed multicultural mathematics in an elementary or secondary school context (see [31] and [28]; the latter has an excellent bibliography), but fewer resources are available at the college level. Students in *Multicultural Mathematics* are expected to not only master the mathematics, but to make comparisons between cultures, to understand the differences between ethnomathematics and academic

[1]Although the Aschers focus on mathematical ideas of nonliterate peoples, I would argue, as D'Ambrosio [9, p. 16] does, that one can extend this definition to all cultural groups. For example, sona drawings (from a nonliterate culture) and Celtic knots (from a literate one) are equally valid as subjects for ethnomathematical study.

mathematics, and to explore the historical and cultural role of mathematics in society. Original sources are used whenever possible. I have not used a textbook for my course, but instead have relied on handouts and a course web site (`http://www.sju.edu/~rhall/Multi`).

Although not all my students have been education majors, multicultural mathematics is an ideal focus for a course for future teachers. Liping Ma's research on mathematics teaching in the United States and China confirms that teachers must possess "profound understanding of fundamental mathematics" in order to teach well [23]. The current *NCTM Principles and Standards* [27, p. 3–4] states that effective teaching also involves "approaching the same problem from different mathematical perspectives," and calls for teaching "mathematics as part of cultural heritage"—in today's classroom, "cultural heritage" should encompass all world cultures. Some topics are chosen to address the NCTM standards for grades K-8; however, each topic is viewed through the lens of world cultures. Ethnomathematical topics such as gelosia multiplication (see question 5.5 below) are becoming common in elementary school curricula, and future teachers would benefit from seeing them in their college courses.

Half of the course covers number systems and arithmetic. The remaining half consists of special topics in mathematics, art, and music (I have varied this part of the course from year to year). Creative projects play a central role in the course. Students complete two projects, one on number systems, and one involving mathematically inspired art or music. The questions included in this article provide examples of assignments and topics for classroom discussion. I also assign more routine exercises. For more details on the course, including handouts, quizzes, essay questions, and assignments, see `http://www.sju.edu/~rhall/Multi`.

2 Number systems and arithmetic

Number systems are representations of the natural numbers by words, gestures, counting devices, and written symbols. Arithmetical algorithms are closely tied to number systems; for example, all our algorithms for multiplication depend on using a positional system. The goals of this unit are to understand and compare four ways of representing numbers (by words, body counting, counting devices, and writing), four systems of writing numbers (additive, alphabetic, multiplicative, and positional), and to understand their role in arithmetical algorithms. We end the section with a discussion of irrational numbers. Students also complete a group project in which they create their own number system (see section 4).

My favorite general resource on spoken and written numbers is Flegg [14, 13], though he emphasizes Western systems. Zaslavsky [32] contains interesting case studies on the role of mathematics in two African cultures, Southwest Nigeria and East Africa, including spoken number systems, finger counting, time-reckoning, number superstitions, and commerce. Although the course is not a history of Western mathematics, we do study the origins of Western counting systems and arithmetical algorithms. Flegg [14, 13] is a good place to start. Menninger [24] is another resource, though the book, originally published in 1957, lacks up-to-date information on some topics. Ifrah [19] is crammed with information and illustrations, but also contains many inaccuracies.[2]

2.1 Number systems

Mathematical concepts tied to counting systems include one-to-one correspondence, base, place value, and order of magnitude. We study several types of representations of numbers (summarized below), starting with the structure of spoken number words. Body counting explains the almost universal preference for base ten, although there are peoples such as the Igbo of Nigeria who do not count in base ten (see question 5.1 below). Counting devices include tokens and counting boards and usually coexist with other ways of representing numbers. Written number symbols are the most familiar representation of numbers, and students should see

[2]For a caveat on Ifrah's "numerous misreadings, misinterpretations, and pure fabrications," consult [10]; the book was published without peer review and ignores much recent scholarship.

at least one example of each of the four types of written systems named below.

When my students studied Kamba and Maasai finger counting [32, pp. 243-251], one asked "why did they need finger counting, when they already had names for numbers?" Multiple representations of the same number abound in our culture, too—think of the phrase "six of one and half a dozen of the other." In addition to the words "six" and "half a dozen", the tally mark ~~||||~~ |, the numeral 6, the Roman numeral *vi*, and holding up six fingers are all commonly used. Students can think of contexts for each representation, and discuss whether we "need" all these ways to represent six. Although Howard Gardner's theory of multiple intelligences [15] doesn't fully explain why we have so many ways to indicate numbers, it is interesting to see how many of the intelligences are used in different types of number systems.

Spoken numbers Spoken number systems connect language with mathematics. The structure of number words is a microcosm of the formal structure in language. The challenge each culture faces is to use a small set of words to represent an infinite (or at least very large) set of numbers. One approach is to have only a few number words. Anything larger is simply called "many." The first numbers in Indo-European were *one* and *two*; the word for *three*, related to *très* ("very" in French), originally meant "many" [24, p. 17]. The use of a base allows number words to be recycled, enabling the representation of much larger numbers[3] (see question 5.1 for an example). We usually compare about fifteen different spoken number systems, so that students see that disparate cultures have solved this problem in similar ways.

Body counting The choice of base ten is evidence of the prevalence of body counting. Zaslavsky [32] illustrates a system from Africa in which spoken numbers are actually descriptions of hand positions. Although we are all used to using our fingers to represent the numbers one through ten, not every body-counting system relies on a one-to-one correspondence between parts of the body and objects counted. The eighth-century English monk and scholar Venerable Bede described a *positional* finger counting system that represented the numbers up to 10,000 on both hands and expressed the numbers up to 1,000,000 with gestures [24, pp. 201-217]. This system dates back to Roman times and prevailed throughout the Middle East and Europe at least to the sixteenth century. Several texts describe its use in recording carries in arithmetic.

Counting and calculating devices Counting devices existed (and still exist) in many cultures, and often these devices influenced the development of spoken or written numbers. Some Yoruba number words [32, p. 206] have their origin in the use of cowries for counting and calculation, and the Chinese "stick numerals" [24, p. 368] are essentially pictures of counting rods. Moreover, some of these devices were the primary means by which merchants and scholars performed calculations. The Inca quipu is a particularly sophisticated example [5], capable of representing not only numbers, but calculations and even equations. The use of a counting board was a precursor to a positional written number system in Europe. The Greek historian Polybius noted

> The courtiers who surround kings are exactly like counters on the lines of a counting board, for, depending on the will of the reckoner, they may be valued either at no more than a mere chalkos, or else a whole talent (Polybius, 2nd century BC, in [14, p. 178]).

In other words, the idea that the order of magnitude of a symbol can vary with its position was already firmly in place, even though a positional written number system was not to be adopted in Europe for another fourteen centuries.

Written numbers Menninger [24] classifies number systems into four types: additive, alphabetic, multiplicative, and positional. *Additive* number systems use one-to-one correspondence to represent quantity.

[3] An extreme example is the Ancient Buddhists, who had names for numbers up to 10^{53}. Their need for such large numbers was religious, rather than practical [24, p. 137].

Each symbol represents a denomination (usually a power of the base), and one adds the values of all the symbols in a number. Egyptian hieroglyphics are the classic example. Even our Hindu-Arabic system has some traces of this principle (try writing $-$, $=$, $\equiv$ quickly, and you'll get 1, 2, 3). *Alphabetic* systems, such as Greek and Hebrew, form a correspondence between numbers and letters of the alphabet, allowing for some curiosities such as chronograms and numerological interpretations.[4] *Multiplicative* systems follow the structure of spoken numbers. They usually have separate symbols for $1, 2, \ldots, b-1, b, b^2, b^3, \ldots$, where b is the base. Further symbols are formed by combining these primary symbols, in the same way that *twenty* is composed of *two* and *ten*. The Greek acrophonic system [19, p. 182], and early Indian numbers [24, p. 395] used this principle. The Hindu-Arabic system we use is a *positional* system, as are the Mayan and Babylonian systems. Positional systems rate highly on the scale of efficiency, number of symbols needed, and ease of computation. They are the only systems that do not (theoretically) require an infinite number of symbols. However, they have little intuitive visual interpretation, little relationship to spoken numbers, and require the invention of a placeholder.[5]

For each written system, we ask the following questions: *What is the base? Why was that base chosen? How easy is the system to learn and to write? Is there any ambiguity in how symbols are interpreted? Which of the other forms of number systems—spoken numbers, body counting, and counting devices—exist in this culture? Do they use the same base? Do they follow the same pattern in other ways?*

2.2 Arithmetic

In this unit, students learn several arithmetical algorithms, compare them, and discuss the dependence of the algorithm on the type of number system used. Addition and subtraction with additive written numbers can be done visually; one merely recopies or erases the symbols, regrouping if necessary. Multiplication is cumbersome; the Egyptians used a system of doubling and adding (also called "Russian peasant multiplication"). For an exhaustive treatment of various algorithms for multiplication, including Egyptian multiplication, multiplication on the abacus, gelosia multiplication, and Vedic multiplication, see Joseph [20, pp. 85-125] (see also the Babylonian multiplication table in question 5.2 and an example of gelosia multiplication in question 5.5). Multiplicative algorithms are sometimes extended to division. The Babylonians calculated tables of reciprocals, so as to divide by multiplying by the reciprocal [21, pp. 102-3]. In addition, most cultures used some sort of calculating device. The Greek word for doing arithmetic—an ancestor of our word *calculator*—originally meant "moving pebbles." There are also some interesting finger tricks for multiplication—see [19, p. 59-61].

At this point, we revisit the number systems introduced earlier in the course to see how amenable the systems are to calculation. Additive systems are—not surprisingly—superb for addition and subtraction, multiplicative systems do reasonably well with multiplication, and positional systems are the most elegant for multiplication and division. Alphabetic systems fare poorly in this comparison—but one should remember that written numbers were not the only representations of numbers used in a culture, and alphabetic numerals were sufficient to record the results of computations on a counting board or abacus.

[4] Some students insist that Roman numerals are alphabetic, because they are letters, or positional, because in numbers such as *"iv"* and *"vi"* the value of *"i"* changes with its position, but, in fact, Roman numerals come from tally marks and form a modified additive system. Normally the values of symbols are added; sometimes they are subtracted.

[5] The history of *gubar numerals* is a good illustration of the resistance to positional numbers. Before the Hindu-Arabic numerals were accepted in Europe, traders in Moorish Spain adopted this system, in which the first nine digits of our system were used. Beyond that, $10 = \dot{1}, 20 = \dot{2}, 100 = \ddot{1}$, etc. [24, p. 416]. Thus, 2078 would be written $\dddot{2}\dot{7}8$. It is curious that the Europeans initially resisted the positional written system, even though they had both a positional finger counting system and a positional counting board. One explanation is that before paper was widely available, written calculations were prohibitively expensive; the positional system's chief advantage is the ease of computation.

2.3 Completing the number line

The development of a system for representing fractions occurred much later than the counting numbers. The Egyptians had a notation for unit fractions (the reciprocals of the counting numbers). All other fractions were written as sums of unit fractions with *different* denominators (for example, $3/8 = 1/4 + 1/8$). This system made computation quite tricky—see [14, p. 131–143] and [18]—but leads to some interesting number theory, including an open problem posed by Erdös and Straus [25, Chapter 30] expored in question 5.4 below.

The Babylonians were the first to develop an analog of our decimals (since they used base sixty, their notation is properly called *sexagesimal*). To the right of the units column are columns for 1/60 ("minutes"), 1/3600 ("seconds"), and so on. Even after the decimal system was adopted, sexagesimals were often preferred for representing fractions, since sixty has more divisors than ten, and they are still used in some contexts. My students are surprised to discover that some repeating decimals have terminating sexagesimal expansion. One can also discuss the implied use of infinite series in repeating decimals. A history of decimals is found in Flegg [14].

Several additional topics can be introduced here: the relationship between rational and irrational numbers, the difference between algebraic and transcendental numbers, and the discovery of negative numbers. The estimation of π has a rich history, dating back at least to the Egyptians. The Ahmes Papyrus [21, p. 82] gives an Egyptian estimate of π. Compare also Gerdes' demonstration of how one can estimate π using African mats [17] and Archimedes' geometric methods.

Square roots also presented problems in ancient mathematics. A Babylonian tablet from 1800-1600 B.C. contains an estimate of $\sqrt{2}$ that is accurate to five decimal places [21, p. 102–6]. Joseph infers that the Babylonians used an iterative method to estimate square roots. On the other hand, Aristotle knew of a proof (see [7, p. 83–84]) that $\sqrt{2}$ is irrational. We discuss the differences between the Greek and Babylonian treatments of irrational numbers, and the meaning of "proof."

3 Special topics in mathematics, art, and music

This part of the course concerns mathematical ideas found in folk arts and music. I vary the topics from year to year. My favorites are the mathematics of drumming and the mathematics of sona drawings, which I describe below. I have also listed some additional topics that can be incorporated into the course. Ascher ([2], [5], and [3]), Gerdes [17] and [16], Zaslavsky [32], and Eglash [12] are good sources of inspiration. Students complete a project in mathematically inspired art or music (see section 4).

3.1 The mathematics of drumming.

Rhythm is a primary component of jazz, popular, and folk music of many cultures, and is central to dance and metrical poetry. Related topics include periodic functions, ratios, combinatorics, visual patterns, and algebra. In addition, the classification of rhythms in poetry led to ancient discoveries of Pascal's triangle and the Fibonacci numbers (see question 5.7). I hold this portion of the class in a computer lab; we use Adlai Waksman's Java program, Abcdrums, to create and hear examples of drumming concepts (see `http://www.sju.edu/~rhall/Multi` for a links to the program, a tutorial, sample compositions, handouts, and class exercises).

There is a strong relationship between drum patterns and geometric concepts, including symmetry, translation, expansion and contraction, self-similarity, inversion, and complementation. It is necessary to use some written rhythmic notation; however, my students do not have to learn to read music. We use the symbol `x` to represent a drum hit and the symbol `.` to denote a rest (so that ♩. ♩. ♩ and ♪𝄾𝄾♪𝄾𝄾♪𝄾 both

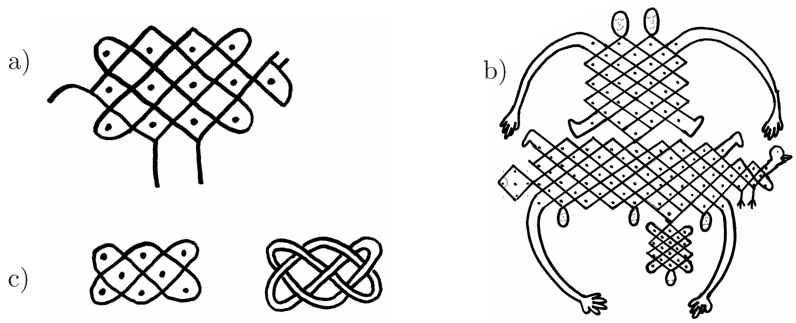

Figure 1: a) Antelope made from basic sona unit. b) *My Family*, by student Maiko Tomioka. c) Relationship between sonas and Celtic knots (and the knot classified as 7_4).

become `x..x..x.`). Abcdrums makes use of the same rhythmic notation, as do some online sources such as the Rhythm Catalog [26] which give instructions for rhythms found in various cultures.

Mathematics can help students discover relationships between rhythms and generate interesting new patterns. Using our notation, the inversion of `x..x..x.` is `.x..x..x`, the complement is `.xx.xx.x`, and a possible augmentation is `x.....x.....x....` Much drumming consists of cyclic patterns—the notation `|: x..x..x. :|` indicates an “infinite” repeat of the same rhythm. The cyclic shifts of this rhythm are `|: .x..x..x :|`, `|: x.x..x.. :|`, etc. Polyrhythms—produced by dividing a measure in two or more different ways simultaneously—are related to fractions and the least common multiple/greatest common divisor. For example, if one drum is to play three equal beats in the same space of time as another drum is to play four equal beats, we must divide the measure up into twelve beats (or a multiple of twelve); one drum plays `x...x...x...` and the other plays `x..x..x..x..` In general, if one drum plays m beats and the other plays n beats, we must divide the measure up into a multiple of $\mathrm{lcm}(m, n)$ beats.

Compositional structure looks a little like elementary algebra. Western music often follows basic structures like AABB or AABA (where A and B represent musical phrases); some Indian classical music has more complex structures incorporating, for example, arithmetic series (AB AAB AAAB AAAAB) or permutations (AAB ABA BAA). Abcdrums allows users to build compositions for up to sixteen different drums by creating patterns (denoted by letters) and then assembling patterns into sequences such as AABA. My students use this “compositional algebra” to write their own drum compositions, as decribed in section 4 below. I require that all the parts have the same number of beats; they have to use some simple algebra to achieve this.

3.2 Sona drawings, Celtic knots, graphs, and knots.

Sona drawings are sand drawings made by the elders of the Chokwe people of South Central Africa. The drawings, which may depict animals or abstract relationships, consist of curves traced around a framework of dots—see Figure 1 for examples. The creator tells a story at the same time as he sketches the picture. With the exception of small lines added as decoration, the curves in most sona drawings are traceable graphs[6]; the topological properties of the graphs sometimes relate to features of the story. Students can explore connections to graph theory. Gerdes [17, p. 156–206] has a wonderful chapter on the mathematical exploration of sona drawings, which also involves some number theory (see question 5.6 for an example). For similar drawings from the Vanuatu of the South Pacific, see [8].

Celtic art abounds with knotwork decorations. A sona drawing—or, indeed, any graph in which all vertices have degree four—is made into a knot by replacing each vertex by a knot crossing. One can show that a knot with alternating crossings can always be formed in this manner. The mathematical field of knot

[6]Note that the student’s drawing (reminiscent of the traditional “lioness and cubs” motif) in Figure 1.b is not traceable. How could one make the graph traceable, while still preserving the overall design?

theory can be used to classify decorative knots.

3.3 Other topics.

Many other topics are available for exploration. Logic puzzles and strategic games are treated in the Aschers' work. Students can draw a directed graph to analyze the game *pong hau k'i* [31], as Ascher does of *mu torere* [1] (see question 5.8). Magic squares, discovered in China and the Middle East, have interesting number theoretic properties [28, p. 50]. Symmetry, including strip patterns, wallpaper patterns, and tilings, is found in art all over the world ([32] and [16]). Fractal geometry is found in African art [12]. We study the derivation of area formulas by dissection methods (including Archimedes' method of estimating π). Students compare several treatments of the Pythagorean Theorem: the Chinese *gou gu* theorem [21], Thabit Ibn Qurra's dissection method, and Euclid's proof, and relate these proofs to geometric designs that suggest the Pythagorean theorem. See Gerdes' discussion of the Pythagorean Theorem in weaving [17]. I also discuss Pythagorean triples, found on the Babylonian Plimpton tablet, and their role in the history of Fermat's Last Theorem. And students can investigate biographies of mathematicians from different cultures—see [22], which compares the mathematical backgrounds of Ramanujan and Hardy.

4 Class projects

I have my students do two projects—each student does a number systems project in a group of three, then the second project (original art or music) is done individually. Projects are graded on creativity, complexity, and use of mathematical concepts we discussed in class or in the readings. They must be original—that is, not a reproduction or variation of traditional artwork or of artwork by another artist. My students also present their projects to the class.

4.1 Number systems project

Among J. R. R. Tolkien's unpublished writings on the lore of his fictitious world, Middle Earth, is a base-12 number system which included both spoken and written numbers [11]. My students create something similar. Each group of students develops an entire number system, complete with spoken and written numbers, a multiplication table, a "history" explaining the origin of their numbers and significance within the fictional culture that produced them, and a mathematical artifact. Examples of "artifacts" my students have created include calendars, recipes, and even a tombstone. They could not use base 2, 5, 10, 12, 20, or 60. The students were graded on the clarity, elegance, and ease of use of their number system; the accuracy of their results; and their overall presentation.

The most interesting aspect of the project is that the students' difficulties and mistakes mirrored the historical development of number systems. For example, one group's base-6 system included a special number for 100, much as languages like Breton use a base-20 system to count to 100, then use the Latin word for 100. Students favored multiplicative written systems, and almost universally avoided positional systems. I'm not sure if this is because positional systems are difficult, or because the requirement that they create a spoken and written system simultaneously made them more likely to create a written system that followed their linguistic structure.

4.2 Drum composition

My students use Abcdrums to write a drum composition, which they present to the class. They use some of the compositional techniques we discuss in class such as polyrhythms, repetition, variations in density, complementary patterns, and expansion or contraction of patterns (the only strict rule is that all parts have the same number of beats). In addition, they notate their composition and write a short essay on the way they used mathematical techniques to create it.

4.3 Sona design and story

The assignment is to create one or more original sona drawings, using traditional sona construction methods, and use them to tell a story. Students should indicate how the drawing is paced to go along with the story. In addition, they write a short essay on the way they used mathematical techniques to create the sona.

5 Questions on selected topics

The following are examples of activities for in-class work and discussion. For more examples, as well as essay questions and drill exercises, see www.sju.edu/~rhall/Multi.

5.1 Counting in Igbo

The following table gives some of the number words in Igbo, a Nigerian language [32, p. 44]. Assume the rest of the Igbo numbers follow the same pattern.

1	otu	6	isii	11	iri na otu	40	ohu abuo
2	abuo	7	asaa	12	iri na abuo	50	ohu abuo na iri
3	ato	8	asato	20	ohu	100	ohu iso
4	ano	9	toolu	21	ohu na otu	300	ohu iri na ohu iso
5	iso	10	iri	30	ohu na iri	400	nnu

To what number does *ohu abuo na iri na ano* refer? What is the Igbo word for 16? For 71? What is the base of the Igbo counting system, and how do you know? What is the secondary base?

The base is 20 and the secondary base is 10. Notice that there is a new word for 400 but not for 100.

5.2 Babylonian archaeology

Translate the Babylonian clay tablet in Figure 2. (from [19, p. 155]). Supply the missing numerals, and find the mistake. What do you think is the purpose of the tablet?

The tablet is a 25 times table. Babylonians used tables for multiplication—in a sexagesimal system, one would theoretically have to memorize up to 59×59 *to do positional multiplication!* This tablet includes only the products of 25 with 1–20, 30, 40, and 50, thus revealing how the Babylonians used their secondary base of 10 to multiply.

5.3 Choosing a base

There are many issues of complexity involved in choosing a base. In the binary system, only two symbols are needed, but numbers grow long quickly. On the other hand, a base-60 system needs 60 symbols, but can represent 3599 using only two digits! Define a "vocabulary" function $v_b(n)$ that equals the number of different symbols needed to count to n base b, and a "length" function $l_b(n)$ that equals the number of symbols needed to represent n. The length $l_b(n)$ has logarithmic growth, while $v_b(n)$ equals n if $n \leq b$

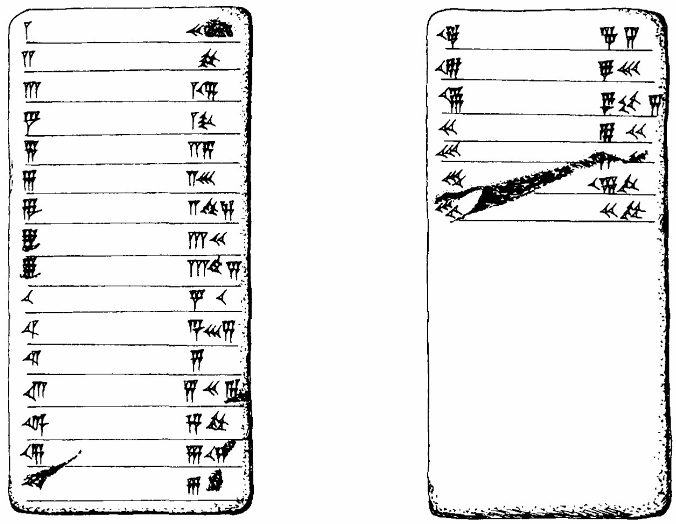

Figure 2: Babylonian tablet, first half of the second millenium B.C. (from [19, p. 155])

and equals b when $n > b$. Using a graph of $l_b(n)$ and $v_b(n)$ for several values of b, students can debate the question: Which range of values of b do you think is best if we want to count to 1,000? to 10,000? The choice of base also affects the amount of memorization required to perform positional multiplication. Divisibility is another issue. Which choices of a base maximize the number of simple fractions which are represented by terminating "decimals"? Which of the following bases do you prefer: 2, 4, 10, 12, 60, or something not on this list, and why?

5.4 Egyptian fractions and Number Theory

Egyptians wrote every fraction as a sum of unit fractions (i.e. reciprocals of counting numbers) with *different* denominators (see section 2.3 for more explanation and examples). This series of questions explores some of the mathematics involved.

1. One way to write any fraction as a sum of unit fractions is the "greedy algorithm": begin by subtracting the largest unit fraction less than your original number, then subtract the largest unit fraction less than the remainder, and continue the process until the remainder is a unit fraction. For example, if the starting fraction is $5/7$, first write $5/7 - 1/2 = 3/14$, then $3/14 - 1/5 = 1/70$, which is a unit fraction. This tells us that $5/7 = 1/2 + 1/5 + 1/70$. Explain why the unit fractions subtracted in this process always have different denominators, and why the greedy algorithm eventually terminates—thus showing that every fraction can be written as a sum of unit fractions with different denominators.

2. It is possible to write every fraction of the form $2/n$, where n is odd, as the sum of two different unit fractions. Since you can't test every possible fraction, what is required to show that this statement is true? Write the fractions $2/5$, $2/7$, $\ldots$, $2/13$ using the greedy algorithm and look for a pattern.

3. It is *not* possible to write every fraction of the form $3/n$ as the sum of two different unit fractions. To prove this, show that $3/7$ cannot be written as the sum of two different unit fractions. (Hint: show that if $3/7 = 1/n + 1/m$, and $1/n$ is larger than $1/m$, then $1/n$ is a unit fraction greater than $3/14$ and less than $1/2$.) However, $3/n$ can always be written as the sum of three different unit fractions. Show this. (Hint: use your answer for the previous question.)

4. The previous questions are examples of a type of mathematics called Number Theory. Discuss your experience answering the questions. Do you find the questions interesting? Challenging? Describe the

thought processes needed to answer these questions. Can you think of other situations in which these logical strategies might be useful?

5. In the 1960's, the mathematicians Erdös and Straus posited that every fraction of the form $4/n$ can be written as the sum of three different unit fractions [25, Chapter 30]. Mathematicians still don't know whether or not their statement, known as the Erdös-Straus Conjecture, is true! What do mathematicians mean by *conjecture*? By *counterexample*? By *unsolved problem*? How many articles about Egyptian fractions can you find on MathSciNet? Can you show the conjecture is true for *some* fractions of the form $4/n$?

These questions show how topics from ancient mathematics (or indeed, ethnomathematics) can inspire professional mathematicians. They also show that mathematics is a dynamic field, not one in which all the problems have been solved.

5.5 Gelosia multiplication

Figure 3 displays an example of gelosia multiplication from Medieval Europe, showing the multiplication 987 $\times$ 961 = 948,507. How does it work? Why does it work? Try 428 $\times$ 790.

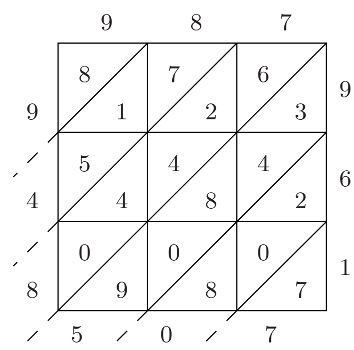

Figure 3: Gelosia multiplication: 987 $\times$ 961 = 948,507

Once the small triangles are filled, addition is done along the diagonal, starting at the lower right and carrying if necessary. It works because the entries on each diagonal have the same place value.

5.6 Sona designs

(This exercise is from Gerdes [17]. This is a sketch only—students will need more guidance in order to solve the problem.) Figure 4 displays the steps involved in the construction of a simple plaited-mat sona. First one draws the dots, then traces a closed loop that weaves around the dots. We will call this the 4×3 plaited mat, since the first step is to draw a lattice of 4 rows and 3 columns of points. This design is complete after only one loop. Explain in your own words how to draw the sona, and draw some sonas of different sizes, such as 2×3 and 4×6. *Students should discover that more than one loop is needed for some sizes, such as* 4×6.

Make a chart of values of r and c from one through ten and fill in the number of loops needed for each pair (r, c) (you should work in groups). If r equals the number of rows and c equals the number of columns in the first step of the design, what is the relationship between the number of loops, r, and c?

The answer is that the number of loops is the greatest common divisor of r and c. For a more in-depth project, analyze one of the other "space-filling" sona patterns in Gerdes in the same way as the plaited-mat

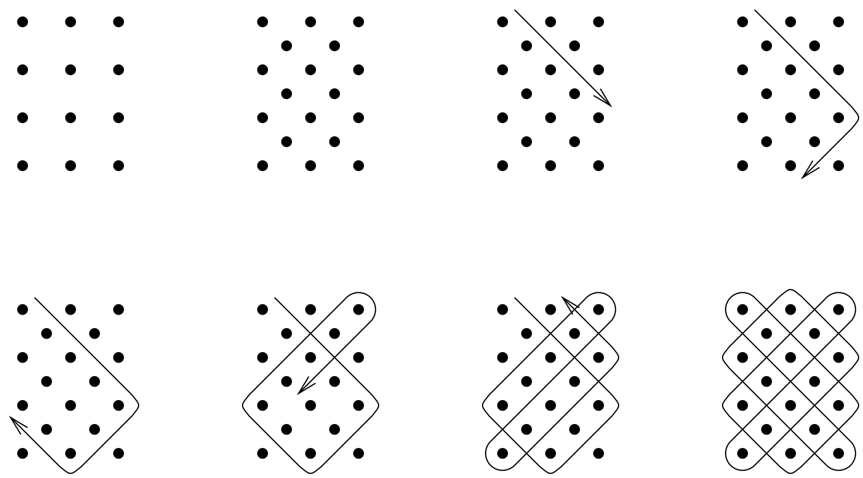

Figure 4: Steps in the construction of a plaited-mat sona

design.

5.7 Classification of poetic meters

Rhythm is closely related to meter in poetry. Syllables in Sanskrit poetry are either long or short, in the ratio 2 : 1. Pingala's Chandahsutra (200 B.C.) considered the problem of which different rhythms can be produced if the number of syllables is fixed [6]. For example, Pingala classified 16 different meters of four syllables in the following way:

1	meter of four short syllables (SSSS)
4	meters of three shorts and a long (LSSS, SLSS, SSLS, SSSL)
6	meters of two shorts and two longs
4	meters of one short and three longs
1	meter of four longs

He proceeds to give a rule for classifying meters of any given number of syllables. Write down the classifications for one, two, three, four and five syllables.

Students should show that the meters of n syllables are classified by the nth row of Pascal's triangle. Note that Pingala lived eighteen *centuries before Pascal!*

There is another type of meter in Sanskrit poetry: meters in which the duration of a cycle is fixed, but the number of syllables is not. The Jain writer Acarya Hemacandra (mid 12th century A.D.) discovered the rule for counting the number of meters of a fixed duration [30]. For example, there are 5 meters of duration 4: SSSS, SSL, SLS, LSS, LL. Find the number of meters of duration 1, 2, 3, 4, and 5. What's the pattern? Can you explain why you get that pattern?

The answer is the Fibonacci numbers. Hemacandra lived a century before Fibonacci. Singh [30] theorizes that Fibonacci may have been influenced by the knowledge of the "Fibonacci" numbers in India.

5.8 Pong hau k'i

The Chinese game *pong hau k'i* is found in [31, p. 187]. Figure 5 shows the game board and some possible positions. An interactive version of the game is found at `http://nrich.maths.org` (keyword: pong). There are two players, each of whom has two counters, with the starting position as shown. Players take turns

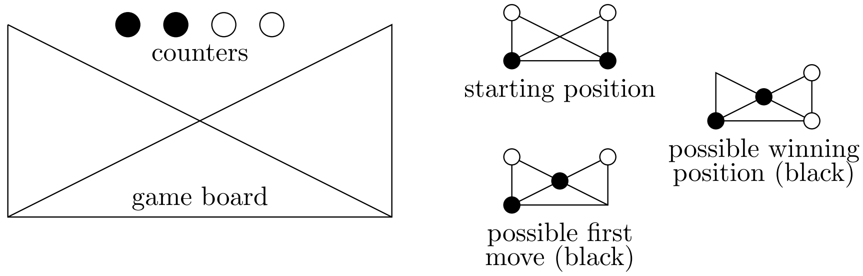

Figure 5: Game board and possible moves for *pong hau k'i*.

moving their counters along one of the lines to an intersection. The object is to block the other player. What is the best strategy?

There is no winning strategy; however, there is a strategy that avoids losing. Students discover this fairly quickly. A more difficult project is to draw a directed graph that represents all possible states of the game, as Ascher does of mu torere *[1]. Analysis of this graph shows that it is possible for either player to win, though one player can win in far fewer moves than the other. If both players know the strategy, the game ends in a draw.*

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I would like to thank my students, whose comments and criticisms have contributed to the design of the class and given me more insight into multicultural mathematics.

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