THE MATHEMATICS OF MUSICAL INSTRUMENTS

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HISTORY

• Bone flutes and whistles were found at Neanderthal sites.

• 9,000-year-old flute is world’s oldest playable instrument.

• Study of the mathematics of music dates back to Pythagoreans.

• Research is still being done today.
THE WILLOW FLUTE
(SELJEFLØYTE)

- a member of the recorder family

- has no finger holes
PITCH AND FREQUENCY

Playing the willow flute causes periodic variations in the air pressure inside the instrument.

The frequency of these variations determines the pitch of the note.

How does the willow flute produce notes of different pitch?
THREE VARIABLES

Suppose

\[ t = \text{time} \]
\[ x = \text{distance along the tube} \]
\[ u(x, t) = \text{pressure at position } x \text{ and time } t \]
THE WAVE EQUATION
IN ONE DIMENSION

The willow flute is modelled by the one-dimensional wave equation

\[ a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \]

where \( a \) is a constant, with two possible sets of boundary conditions:

\[ u(0, t) = 0 \quad u(L, t) = 0 \quad \text{(end open)} \]
\[ u(0, t) = 0 \quad u_x(L, t) = 0 \quad \text{(end closed)} \]

where \( L \) is the length of the flute and we choose units such that the outside pressure is 0.
If the end of the pipe is open, solutions to the wave equation are linear combinations of solutions of the form

$$u(x, t) = \sin \left(\frac{n\pi x}{L}\right) \left(b \sin \left(\frac{a_n \pi t}{L}\right) + c \cos \left(\frac{a_n \pi t}{L}\right)\right)$$

where $n = 1, 2, 3, \ldots$, and $b$ and $c$ are constants.
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where \( n = 1, 2, 3, \ldots \), and \( b \) and \( c \) are constants.

Fix \( n \) and \( x \) and vary \( t \). The pressure varies periodically with period \( 2L/an \). Therefore,

$$\text{frequency} = \frac{an}{2L}$$

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for \( n = 1, 2, 3, \ldots \).

If the end of the pipe is closed, the possible frequencies are

\[
\text{frequency} = \frac{an}{4L}
\]

for \( n = 1, 3, 5, \ldots \).
Since frequency equals $\frac{an}{2L}$ for $n = 1, 2, 3, \ldots$ or $\frac{an}{4L}$ for $n = 1, 3, 5, \ldots$, we can change the pitch by

- changing the length ($L$) as in the slide whistle. This produces continuous differences in pitch. Cutting holes in the tube creates discrete differences in pitch.

- jumping to a different solution of the wave equation by blowing with more or less force. That is, changing the value of $n$. This causes discrete differences in pitch.

- changing the boundary condition. The change in pitch can be discrete or continuous.
THE OPEN-END SCALE

end open

frequency = \( \frac{an}{2L} \)

\( n = 1, 2, 3, \ldots \)

1:1

1:2

1:3

\[ \text{\red} = \text{Pitches produced with end open} \]
THE COMBINED SCALE

end open

frequency $= \frac{an}{2L}$

$n = 1, 2, 3, \ldots$

end closed

frequency $= \frac{an}{4L}$

$n = 1, 3, 5, \ldots$

1:1

2:1

1:2

2:3

1:3

2:5

---

= Pitches produced with end open

= Pitches produced with end closed
THE MAJOR TRIAD

A major triad is comprised of three notes whose ratio of frequencies (give or take an octave) is 4:5:6.

It is a fundamental building block of Western music.
JUST TEMPERAMENT

The just scale is constructed from three major triads.

<table>
<thead>
<tr>
<th>just temperament</th>
<th>willow flute</th>
</tr>
</thead>
<tbody>
<tr>
<td>1:1</td>
<td>1:1</td>
</tr>
<tr>
<td>9:8</td>
<td>9:8</td>
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<tr>
<td>5:4</td>
<td>5:4</td>
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<tr>
<td>4:3</td>
<td>11:8</td>
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<td>3:2</td>
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<tr>
<td>5:3</td>
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</tr>
<tr>
<td>15:8</td>
<td>7:4</td>
</tr>
<tr>
<td>2:1</td>
<td>15:8</td>
</tr>
</tbody>
</table>

(Chart showing ratios for just temperament and willow flute)
PYTHAGOREAN TEMPERAMENT

Pythagorean temperament is based on octaves and fifths.

The 2 : 1 octave relationship gives a natural equivalence on frequencies $x$ and $y$:

$$x \sim y \Leftrightarrow \log_2 x = \log_2 y \pmod{1}.$$  

For example, $\log_2 880 = 1 + \log_2 440$ so the notes with frequencies 880 and 440 ($A'$ and $A$) are equivalent.
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On the logarithmic scale, using the octave identification, we get

$$\log_2 x \rightarrow \log_2 x + \log_2 \frac{3}{2} \pmod{1}$$

which corresponds to a rotation by $\log_2 \frac{3}{2}$ of the circle formed by identifying the endpoints of $[0, 1]$.  

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Since \( \log_2 \frac{3}{2} \) is irrational, the images of the initial point will form a dense subset of the circle.
Imagine a string attached to infinity extending to the origin along the line $y = \log_2 \frac{3}{2} x$. A nail is driven through each point in the plane with positive integer coordinates. If you pull the free end of the string up or down, it touches the nails that are closest to the line.
APPROXIMATING $\log_2 \frac{3}{2}$ BY A RATIONAL NUMBER

The nails touched by the string give a sequence of increasingly better rational approximations to $\log_2 \frac{3}{2}$. The values are

$$1, \frac{1}{2}, \frac{3}{5}, \frac{7}{12}, \frac{24}{41}, \frac{31}{53}, \frac{179}{306}, \ldots$$
The just-tempered scale is shown in red.

The twelve-note scale in black on the left is called equal temperament and is the system by which pianos are tuned.

What are its advantages and disadvantages? Why don’t we use the 41-note scale?
DRUMS AND BELLS

The behavior of drums and bells is governed by the 2- and 3-dimensional wave equations with various boundary conditions.

In general, the modes of vibration are not harmonic. That is, they are not rationally related to the fundamental.

The famous question *Can one hear the shape of a drum?* asks if two different boundary conditions can produce the same modes of vibration. The problem was unsolved for twenty-four years.
The answer is no.