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Planetary Motion and the Duality of Force Laws*

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Abstract. Trajectories of Hooke's law in the complex plane, which are conic sections, are mapped onto trajectories of Newton's law of gravitation via the transformation $z \to z^2$. Newton's law of ellipses (objects attracted to a center by a force inversely proportional to the square of the distance travel in conic sections) follows from a geometric analysis of this map. An extension of this approach reveals a similar relation between more general pairs of power laws of centripetal attraction. The implications of these relations are discussed and a Matlab program is provided for their numerical study. This material is suitable for an undergraduate complex analysis class.

Key words. two-body problem, functions of a complex variable, geometric function theory

AMS subject classifications. 70F05, 30A99, 30C99

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1. Introduction.

We begin with two seemingly unrelated systems: the orbit of a planet around the sun and the path of a ball swung on a spring. The fact that both the planet and the ball travel in an ellipse is well known. Johannes Kepler was the first to show that the planets travel in ellipses with the sun at one focus. The elliptical path of the ball is easily observable. In this case, the fixed end of the spring will be at the center of the ellipse. Freshman physics tells us that the first system is an example of Newton's law of gravitation (force is inversely proportional to the square of the distance) and the second is an example of Hooke's law (force is directly proportional to distance), and that paths of objects attracted to a center under either law are conic sections.

What is more surprising is that the two orbits are directly related. We will show in this article that Hooke's law and Newton's law correspond by the complex transformation $z \to z^2$. Two laws of centripetal attraction, such as Hooke's law and Newton's law, will be called dual if a complex transformation $z \to z^\beta$ takes the trajectories of one law to the trajectories of the other. We will use complex analysis to discover which other laws are dual. A fascinating footnote to this discussion is that Newton actually knew that his law of gravitation and Hooke's law are dual (though he defined duality without using complex analysis) and was aware of several other pairs

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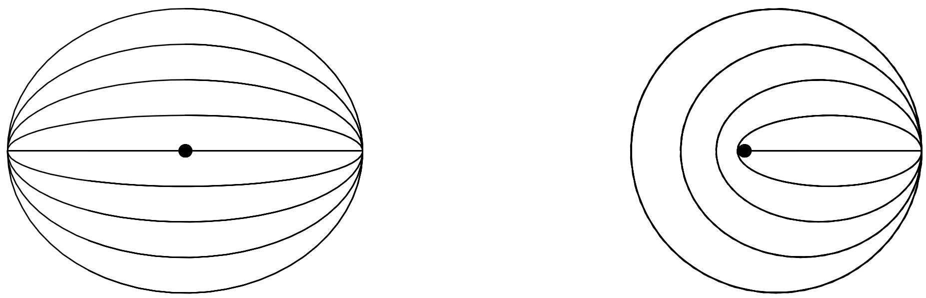

Fig. 2.1 *Trajectories for $w'' = -w$ and their images under the map $z \to z^2$. Note that the degenerate orbit is mapped to an orbit which "bounces off" the origin.*

of dual laws [5, pp. 114–125]. We will examine some observations he made about the bizarre occurrences possible in a universe subject to a different law of gravity.

We begin the discussion by showing that Hooke's law and Newton's law are dual, which is the main result of the paper. A geometric analysis of the map $z \to z^2$ proves that orbits of planets under Newton's law of gravitation are conic sections. The duality between Hooke's law and Newton's law can be extended to all power laws and used to understand some features of trajectories obeying these laws. The Appendix contains a Matlab program which plots dual trajectories. The body of this article is based on presentations by Arnold [4] and Saari [9]. Needham also presents a beautiful, purely geometrical approach to the same problem [8].

2. Hooke's Law and Newton's Law. Karl Sundman (1873–1949) (re)discovered the duality of Hooke's law

$$w'' = -Cw$$

and Newton's law

$$z'' = -\tilde{C}\frac{z}{|z|^3}$$

when he tried to find a complex transformation that would remove the singularity present in Newton's law when $z = 0$.[1] He noticed that nonsingular orbits coming very near the origin spin around it once and are ejected in a direction close to the incoming one, as in Figure 2.1 (right). If one extends the singular orbits (that is, those which are on a collision course with the sun) coming towards the origin by treating the planets as bouncing balls that hit the sun and return back along their incoming paths, then these orbits correspond to images of lines passing through the origin under the map $z \to z^2$. Using this convention Sundman then showed that every orbit under Newton's law was the image of a Hooke's law orbit under the squaring map and was able to find a series solution for the three-body problem [9]. Figure 2.1 suggests Sundman's motivation for studying the squaring map.

Before proving Sundman's result we need to consider one detail. Theorem 2.1, which is the main result of this paper, shows that if $w(t)$ is a path in the complex plane satisfying Hooke's law, the image of $w(t)$ under the map $z \to z^2$ will satisfy

[1] The discovery of this relation is attributed to either Bohlin or Sundman, although in a certain form it was already known to Newton [5]. Arnold [4] attributes the idea of the proof to Bohlin and Saari [9] attributes it to Sundman.

Newton's law. Notice that a crucial piece of information is missing from the previous sentence! A path in the complex plane can be parametrized in many different ways. Even if one parametrization satisfies a given differential equation, there will always be other parametrizations that will not satisfy it.

Which parametrization will work for our purposes? Suppose $w(t)$ is a solution to Hooke's law and $z(\tau) = w^2(t)$ for some parametrization of time $\tau(t)$. We can express $w(t)$ in polar coordinates as $w(t) = (r, \theta)$. Then $z(\tau) = w^2(t) = (r^2, 2\theta)$. Let A_1 and A_2 be the areas swept out by $w(t)$ and $z(\tau)$, respectively. Newton showed that the law of areas must hold in any central field, meaning that $\frac{dA_1}{dt}$ and $\frac{dA_2}{d\tau}$ are both constants if $z(\tau)$ satisfies Newton's Law. Assuming that w does not lie on a line through the origin,

$$\text{constant} = \frac{\frac{dA_1}{dt}}{\frac{dA_2}{d\tau}} = \frac{r\theta\frac{dr}{dt} + \frac{1}{2}r^2\frac{d\theta}{dt}}{2r^3\theta\frac{dr}{d\tau} + r^4\frac{d\theta}{d\tau}} = \frac{1}{2}\frac{d\tau}{dt}\frac{1}{r^2} = \frac{1}{2}\frac{d\tau}{dt}\frac{1}{|w|^2},$$

using the expression for area in polar coordinates ($A = \frac{1}{2}r^2\theta$). Therefore the parametrization $\tau(t)$ must satisfy

$$\frac{d\tau}{dt} = k|w|^2$$

for some constant k. We will choose $k = 1$ for simplicity. If w lies on a line through the origin, we will consider w a limit of nondegenerate trajectories and use the same parametrization. We are now ready to prove the duality of Hooke's law and Newton's law.

THEOREM 2.1. *Suppose the motion of a point in the complex plane is given by $w(t)$ and satisfies Hooke's law $w'' = -Cw$. Then a point following the trajectory $z(\tau(t)) = [w(t)]^2$, where $\frac{d\tau}{dt} = |w|^2$, moves according to Newton's law of gravitation:*

$$\frac{d^2z}{d\tau^2} = -\tilde{C}\frac{z}{|z|^3},$$

where $\tilde{C} = 2(|w'(0)|^2 + C|w(0)|^2)$.

Proof. The proof is a matter of using the chain rule.

$$\begin{aligned}
\frac{d^2z}{d\tau^2} &= \frac{1}{|w|^2}\frac{d}{dt}\left(\frac{1}{|w|^2}\frac{dw^2}{dt}\right) \\
&= \frac{2}{w\overline{w}}\frac{d}{dt}\left(\frac{1}{\overline{w}}\frac{dw}{dt}\right) \\
&= -\frac{2}{w\overline{w}^3}\frac{dw}{dt}\frac{d\overline{w}}{dt} + \frac{2}{w\overline{w}^2}\frac{d^2w}{dt^2} \\
&= -\frac{2}{w\overline{w}}\left[\overline{w}^{-2}\frac{dw}{dt}\overline{\left[\frac{dw}{dt}\right]} + C\frac{w}{\overline{w}}\right] \\
&= -2w^{-1}(\overline{w})^{-3}[|w'|^2 + C|w|^2].
\end{aligned}$$

Now let $E_w = \frac{1}{2}(|w'|^2 + C|w|^2)$, and note that differentiation shows that E_w is constant on any trajectory. In physical terms, E_w is called the *energy* of the harmonic oscillator and is constant along the trajectories of motion because no friction is present to dissipate it. Then

$$\frac{d^2z}{d\tau^2} = -4E_w w^{-1}\overline{w}^{-3} = -4E_w\frac{z}{|z|^3},$$

which is what we intended to show. ☐

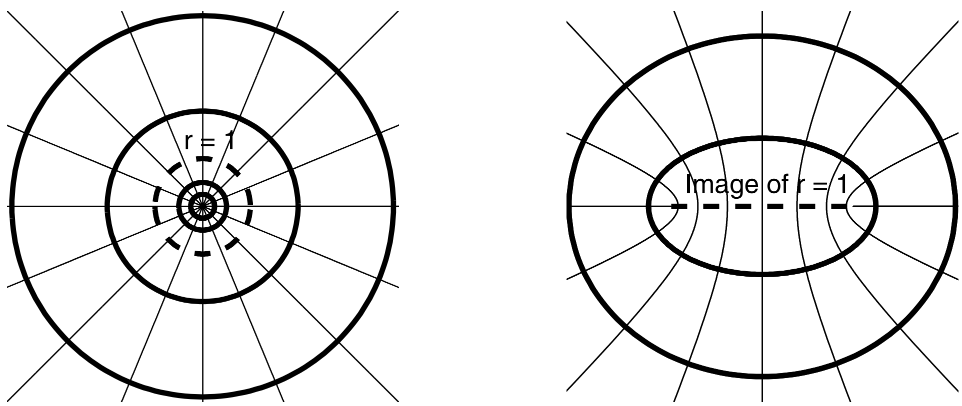

Fig. 3.1 *The transformation $z \to z + \frac{1}{z}$.*

So $z \to z^2$ maps those parametric curves $w(t)$ which satisfy the equation $w'' = -Cw$ into parametric curves satisfying the equation $z'' = -\tilde{C}\frac{z}{|z|^3}$ after a suitable reparametrization of time. In order to understand the implications of this duality, we will consider some complex geometry.

3. The Geometry of the Map $z \longrightarrow z^2$. Now that we know that Hooke's law and Newton's law are dual, let's see what this means in terms of the geometry of orbits. With a little work, one can show that trajectories in the complex plane satisfying Hooke's law are ellipses in the case $C > 0$, lines when $C = 0$, and hyperbolas when $C < 0$ (more precisely, one branch of a hyperbola). The elliptic and hyperbolic trajectories are centered at the origin. Now we will investigate the effect that the transformation $z \to z^2$ has on conic sections. The so-called Zhukovskii ellipses and hyperbolas will be useful in our discussion. Their definition is contained in the following lemma.

LEMMA 3.1. *The transformation $T(z) = z + 1/z$ maps circles centered at the origin two-to-one and onto ellipses with foci at* ± 2 (Zhukovskii ellipses) *and lines through the origin two-to-one and onto hyperbolas with foci at* ± 2 (Zhukovskii hyperbolas).

Proof. Suppose $z = \rho e^{i\theta}$. Then

$$\mathrm{Re}(T(z)) = x = \left(\rho + \frac{1}{\rho}\right)\cos\theta,$$
$$\mathrm{Im}(T(z)) = y = \left(\rho - \frac{1}{\rho}\right)\sin\theta.$$

Eliminating θ by squaring both sides of these equations and adding them leads to

$$\frac{x^2}{\left(\rho + \frac{1}{\rho}\right)^2} + \frac{y^2}{\left(\rho - \frac{1}{\rho}\right)^2} = 1.$$

A circle of radius R centered at the origin is described in polar coordinates by fixing $\rho = R$. Therefore, its image is an ellipse with foci at $\pm\sqrt{(R + \frac{1}{R})^2 - (R - \frac{1}{R})^2} = \pm 2$, except in the degenerate case, $R = 1$, when the image is the line $[-2, 2]$ (see Figure 3.1). Notice that T takes circles of radius R and $1/R$ to the same ellipse, so the transformation is two-to-one.

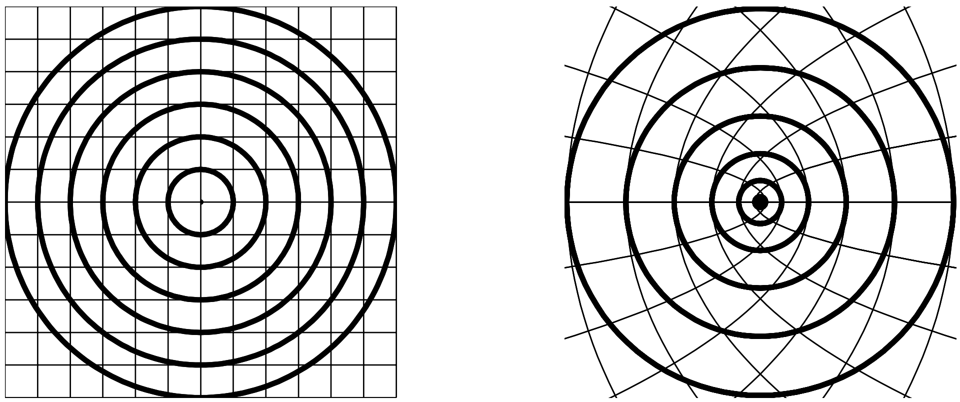

Fig. 3.2 *The transformation $z \to z^2$ acting on circles and lines.*

Eliminating ρ in a similar way gives

$$\frac{x^2}{4\cos^2\theta} - \frac{y^2}{4\sin^2\theta} = 1.$$

A line through the origin has a fixed angle $\theta = \Theta$ (in polar coordinates). Its image under T is therefore one branch of a hyperbola with foci at ± 2. □

The map $T(z) = z + 1/z$ is known as the Zhukovskii map and has applications in fluid dynamics [1].

LEMMA 3.2. *The map $z \to z^2$ takes a Zhukovskii ellipse to an ellipse with one focus at the origin. It takes a Zhukovskii hyperbola to one branch of a hyperbola that has the origin as its focus, and a line to a parabola with focus at the origin.*

Proof. The map $z \to z^2$ takes circles of radius R centered at the origin to circles of radius R^2 centered at the origin, and lines through the origin to rays from the origin. Since $(z + 1/z)^2 = z^2 + (1/z)^2 + 2$ it follows that $(T(z))^2 = T(z^2) + 2$. Therefore $z \to z^2$ acts on Zhukovskii ellipses and hyperbolas as claimed.

The squaring map takes a line parallel to the real axis $z = x_0 + iy$ to a parabola in the complex plane with equation $x = x_0^2 - (y/2x_0)^2$, as shown in Figure 3.2. Any other line in the complex plane can now be obtained from a line parallel to the x-axis by a rotation. Since $[e^{i\theta}(x_0 + iy)]^2 = e^{i2\theta}[(x_0 + iy)]^2$ this completes the argument. □

Since any ellipse or hyperbola with foci equidistant from the origin is obtained by dilation and rotation of a Zhukovskii ellipse or hyperbola, $z \to z^2$ maps all such ellipses to ellipses with one focus at the origin and all such hyperbolas to one branch of a hyperbola with one focus at the origin.

Recall that the type of conic section formed by orbits satisfying Hooke's law is determined by the sign of C. Therefore, C also determines the type of conic section in the dual, Newton's law. In the case of orbits satisfying Newton's law, the energy of a unit mass is $E_z = \frac{1}{2}|z'|^2 - \frac{\tilde{C}}{|z|}$. Using the initial condition for the dual equation and the fact that the energy E_z is constant along any solution, we obtain

$$E_z = 2\frac{|w'(0)|^2}{|w(0)|^2} - \frac{2|w'(0)|^2 + 2C|w(0)|^2}{|w(0)|^2} = -2C.$$

4. Completeness of Theorem 2.1. Theorem 2.1 is quite attractive since it provides a correspondence between solutions to a nonlinear problem (Newton's law) and a linear problem (Hooke's law). Nonlinear problems are typically much more difficult than linear problems. But can we obtain *all* solutions of Newton's law by applying the squaring map to solutions of Hooke's law? We will show that the answer to this question is yes, at least for nonsingular orbits.

A triple (w_0, w_0', C) consisting of initial conditions plus a choice of constant C for Hooke's law uniquely determines a triple $(z_0, z_0', \tilde{C})$ giving initial conditions and the constant $\tilde{C}$ for the inverse square law by

$$(z_0, z_0', \tilde{C}) \;=\; \left(w_0^2,\ \frac{2w_0}{|w_0|^2}w_0',\ 2(|w_0'|^2 + C|w_0|^2)\right).$$

The mapping $(w_0, w_0', C) \to (z_0, z_0', \tilde{C})$ is onto and two-to-one on the initial conditions (except where $w_0 = 0$, where it is undefined) and onto the values for $\tilde{C}$. Consequently, any nonsingular orbit satisfying Newton's law of gravitation is an image of a curve satisfying Hooke's law under the squaring map.[2]

The situation of singular orbits is more problematic. The map $z \to z^2$ takes differentiable orbits to nondifferentiable orbits. Since any singular orbit of Newton's law is the image of an orbit w which lies on a line through the origin and can be brought to the real axis by rotation, we need only discuss real solutions to $w'' = -Cw$. The solution will be either of the form

$$w(t) = Kt \tag{1}$$

for $C = 0$ or of the form

$$w(t) = K\sin(\sqrt{C}t - \delta) \tag{2}$$

for $C > 0$, where K and δ are constants depending on the initial conditions. Under the map $z = w^2$ these orbits are mapped to a portion of the positive real line. However, the parametrization is not smooth, since for real w,

$$\frac{dz}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt}\,w^2 = \frac{dt}{d\tau}\cdot 2w\,w' = \frac{2w'}{w}.$$

If w is given by either (1) or (2), the speed diverges to infinity. This singularity is reached in finite time.

Sundman's convention that the object is reflected at the origin is the natural solution. This can be motivated in two ways: first, this is the limiting case as the orbits become more and more eccentric, and second, if we consider the images of the points of w under the squaring map regardless of the singularity, z changes direction at the origin.[3] With this convention, we can express every orbit of Newton's law as an image of a Hooke's law orbit.

5. Newton's Law of Ellipses. In conjunction with Lemma 3.2 this observation proves Newton's law of ellipses.

[2]Since (w_0, w_0') and $(-w_0, -w_0')$ are mapped to the same point, the squaring map is one-to-one on elliptical orbits (although it is two-to-one pointwise). However, in the case of the nonclosed orbits, (w_0, w_0') and $(-w_0, -w_0')$ lie in different trajectories (either in parallel lines or the different branches of a hyperbola), and the squaring map is two-to-one on these orbits.

[3]A different view of the problem is given by Milnor [7]. He claims that if we graph $z(t)$ (which is a *real* variable here) as a function of time, it satisfies the equation $(z')^2 = 2E + C/r$ and is in fact a cycloid.

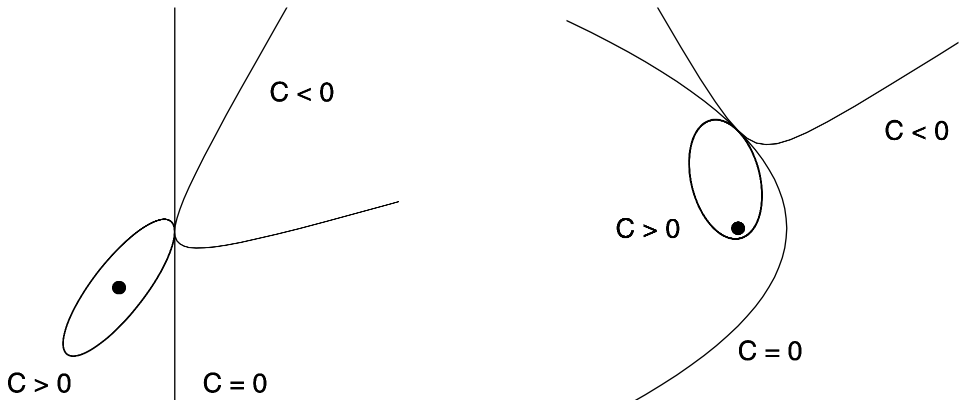

Fig. 5.1 *Trajectories of Hooke's law and their images under $z \to z^2$.*

THEOREM 5.1. *All trajectories of motion satisfying Newton's law $z'' = -\tilde{C}\frac{z}{|z|^3}$ are conic sections. In particular, they are elliptical when $E_z < 0$, hyperbolic when $E_z > 0$, and parabolic when $E_z = 0$.*

There is a slight subtlety in the case of the hyperbolic orbits. For any $\tilde{C}$, hyperbolic solutions to $z'' = -\tilde{C}\frac{z}{|z|^3}$ consist of only one branch of a hyperbola. The choice of branch depends on whether the origin is attractive ($\tilde{C} > 0$) or repulsive ($\tilde{C} < 0$).

See Figure 5.1 for an illustration of the duality for different values of C. This figure was created using the Matlab routines given in the Appendix.

6. Duality of General Power Laws. As it is often the case in mathematics, a particular example is an instance of a more general rule. In our case, we can expand the relation between Hooke's law and Newton's law to relations on all power laws with the following theorem.

THEOREM 6.1. *Trajectories of points in the complex plane under the centripetal attraction $w'' = -C\frac{w}{|w|^{1-a}}$ are mapped to the trajectories of $z'' = -\tilde{C}\frac{z}{|z|^{1-A}}$ under the transformation $z = w^\beta$, where a, A, and β satisfy*

$$(a+3)(A+3) = 4, \qquad \beta = \frac{a+3}{2}.$$

Proof. The law of areas holds in any central field, and so we reparametrize time as

$$\frac{d\tau}{dt} = \frac{|z(\tau(t))|^2}{|w(t)|^2} = |w(t)|^{2(\beta-1)}.$$

The remaining calculations follow those of Theorem 2.1. □

The duality of Newton's law and Hooke's law now becomes a special case of this observation, where $a = 1$, $A = -2$, and $\beta = 2$.

7. Alternative Universes. We will use Theorem 6.1 to investigate a bizarre occurrence possible in a universe subject to the inverse fifth power law of gravity. The so-called self-dual laws (that is, the laws of motion that are transformed into themselves under $z \to z^\beta$) are those for which $a = -1$ or $a = -5$. If $a = -1$, then $\beta = 1$

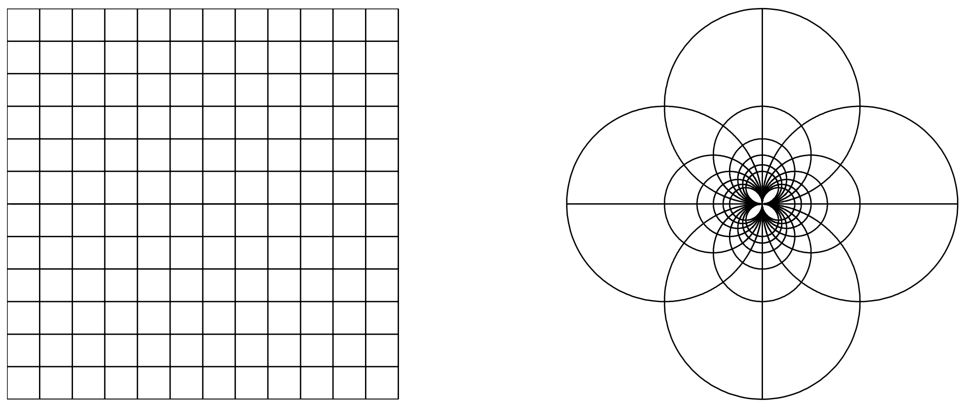

Fig. 7.1 *The transformation $z \to \frac{1}{z}$ acting on lines.*

and we cannot get any information from the duality. However, in the case $a = -5$ we have $\beta = -1$ and the trajectories of motion for the inverse fifth power law are transformed to trajectories for the same law by the inversion $z \to \frac{1}{z}$.

If we set $C = 0$, all trajectories of the inverse fifth power law $w'' = -C\frac{w}{|w|^6}$ are straight lines. Their images under the transformation $z = w^{-1}$ are also trajectories for the same law, since it is self-dual, although the values of $\tilde{C}$ will vary depending on initial conditions. Figure 7.1 shows how inversion acts on horizontal and vertical lines in the complex plane.

We have now shown that a field whose central attraction decreases as the fifth power of the distance contains circular trajectories that pass through the origin. Planets on such orbits would be on a collision course with the sun! This is an instance of a collision singularity, a phenomenon that has been much studied in celestial mechanics [6]. Newton was the first to make this observation [5, pp. 114–125].

Interestingly, all the bounded orbits that we have discussed so far are periodic. For general power laws this will typically not be the case, as you may check by playing with the Matlab program provided. An explanation is given in [3]. Theorem 6.1 also suggests that the inverse third power law is special in some way. What do its trajectories look like?

8. Applications. All solutions of the system $w'' = -Cw$ for $C > 0$ are ellipses, and if we change the initial values of a solution by a little, we obtain a different solution that remains close for all time to the original one. Such systems are called *stable*. An approximate solution to a stable system (found on a computer) will reflect the properties of the real solution. On the other hand, small differences in the initial conditions of *unstable* systems will typically lead to a divergence of orbits in time. Newton's law describes an unstable system. Therefore, the transformation described in Theorem 2.1 takes a stable system to an unstable system, allowing us to find solutions to the unstable system numerically.

Since we do not live in a two-dimensional universe, this approach is of limited practical value. However, if Newton's law and Hooke's law were dual in three dimensions, we could find solutions to problems that involve perturbation of the three-dimensional two-body problem (for example, a satellite orbiting earth) computationally. It appears that the great Italian mathematician Levi-Civita unsuccessfully tried to find such an

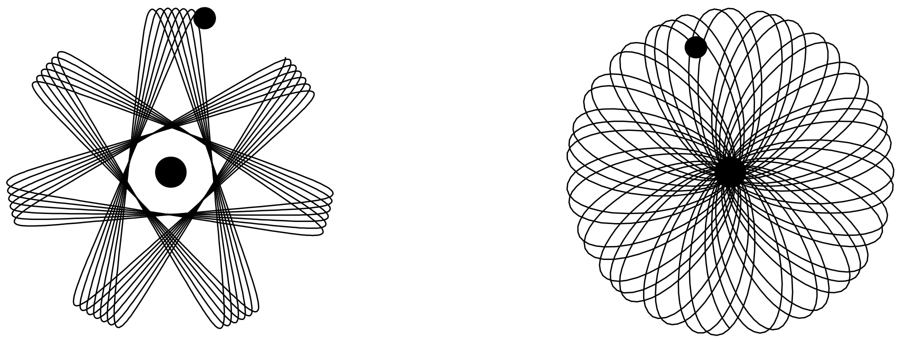

Fig. A.1 *A trajectory for the ninth-power law of attraction and its dual.*

extension. The duality was shown by Kustaanheimo using *spinors*, which are a natural generalization of complex variables [9].

Appendix A. Illustrating the Duality with Matlab. The following Matlab programs will generate pictures of dual trajectories. To use it, download the files from http://www.math.psu.edu/hall/newton/ or copy the program given here, save the first file as `duality.m`, and save the second as `orbit.m` in some directory. Open Matlab in that directory and type `duality` at the `>>` prompt. This will give you a picture for the default values, which illustrates the solution to $w'' = -w$ with initial conditions $w_0 = 1 + i,\ w_0' = -i$ for $0 \le t \le 50$ and the image of that trajectory under the map $z \to z^2$. To change the constants, type `duality(`a`,`C`,`t`,`Y_0`)`, where a and C are as given in Theorem 6.1, t is the maximum time, and Y_0 is the vector `[`$x_0\ x_0'\ y_0\ y_0'$`]`. For instance, `duality(1, -1, 5, [1 0 -1 -1])` gives the duality for Hooke's law ($a = 1$) with a different value for C and a different set of initial conditions. You need not enter all four arguments; `duality(-3, -1)` will just give the inverse cube law with $C = -1$. Some problems you will notice in the program are due to the exponential function. Try `duality(-2)` and see what you get!

You will notice that some laws of attraction have strange, spirograph-type trajectories. For instance, Figure A.1 shows a trajectory for an ninth-power law with $C = 1$. For more information on laws that have only closed orbits and a more or less explicit solution to the laws of motion in a central field, see [3, pp. 33–42].

You can read more about this program at http://www.math.psu.edu/hall/newton/, along with several articles about Newton's laws and other topics from the vantage point of first-year calculus.

Save this file as `duality.m`:

```
function duality(a, C, t, Y0)          % Y0 = [x0 x'0 y0 y'0]
global dualityfig

if nargin < 4, Y0 = [1 0 1 -1]; end
if nargin < 3, t = 50; end             % these are the
if nargin < 2, C = 1; end              % default values
if nargin < 1, a = 1; end

dualityfig = figure( ...
'Name','Dual Trajectories',...
```

```
'Userdata',[a C]);

options = odeset('RelTol',1e-4,'AbsTol',[1e-4 1e-4 1e-4 1e-4]);
       [T,Y] = ode45('orbit',[0 t],Y0,options);

X = Y(:,1);
Y = Y(:,3);

Z = X + i*Y;
x = real(Z.^((a+3)/2));
y = imag(Z.^((a+3)/2));

subplot(1,2,1)
plot(X,Y)
line('Marker', '.', 'Markersize', 35, 'xdata', 0, 'ydata', 0);
axis('square')

subplot(1,2,2)
plot(x,y)
line('Marker', '.', 'Markersize', 35, 'xdata', 0, 'ydata', 0);
axis('square')
```

Save this file as `orbit.m`:

```
function dy = orbit(t,y)
dualityfig = gcf;
A = get(dualityfig,'UserData');
a = A(1);  C = A(2);
dy = zeros(4,1);
dy(1) = y(2);                                   % y(1) = x
dy(2) = -C*y(1)/(y(1)^2 + y(3)^2)^((1-a)/2);    % y(2) = x'
dy(3) = y(4);                                   % y(3) = y
dy(4) = -C*y(3)/(y(1)^2 + y(3)^2)^((1-a)/2);    % y(4) = y'
```

REFERENCES

[1] D. J. Acheson, *Elementary Fluid Dynamics*, Clarendon Press, Oxford, UK, 1990.
[2] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill, New York, 1979.
[3] V. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1989.
[4] V. Arnold, *Huygens and Barrow, Newton and Hooke*, Birkhäuser, Boston, 1990.
[5] S. Chandrasekhar, *Newton's* Principia *for the Common Reader*, Clarendon Press, Oxford, UK, 1995.
[6] F. Diacu and P. J. Holmes, *Celestial Encounters: The Origin of Chaos and Stability*, Princeton University Press, Princeton, NJ, 1996.
[7] J. Milnor, *On the geometry of the Kepler problem*, Amer. Math. Monthly, 90 (1983), pp. 353–365.
[8] T. Needham, *Newton and the transmutation of force*, Amer. Math. Monthly, 100 (1993), pp. 119–137.
[9] D. G. Saari, *A visit to the Newtonian n-body problem via elementary complex variables*, Amer. Math. Monthly, 97 (1990), pp. 105–119.