The Sound of Numbers
A Tour of Mathematical Music Theory

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**Introduction**  

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Preface

Abstract

The Sound of Numbers is a book on the mathematics of music theory—that is, the use of mathematics to describe, analyze, and create musical structures such as rhythms, scales, chords, and melodies. Music theorists have used mathematics to solve musical problems for centuries. Mathematicians, too, have investigated musical questions. Some composers have turned to mathematics for inspiration. However, there has been a significant disconnect between these two fields since the eighteenth century. This book aims to interpret the work of music theorists in a manner that is accessible to readers with scientific training and some background in music, and it aims to explain how problems in music theory are connected to problems that arise in mathematics.

Introduction

There are a surprising number of applications of mathematics to music theory. Questions about variation, similarity, enumeration, and classification of musical structures have long intrigued both musicians and mathematicians. In some cases, these problems inspired mathematical discoveries. In ancient India, for example, investigation of rhythm led to the discovery of the Fibonacci numbers and Pascal’s triangle. Pages from Euler’s notebooks reveal that he experimented with tuning systems and rhythmic variation. Although this book does not describe groundbreaking mathematical results, it nonetheless contains problems that are appealing, nontrivial, and, in some cases, connected to deep mathematical questions.

The Sound of Numbers covers topics that are interesting from both a mathematical and musical standpoint. These include techniques discovered in ancient India for counting and
creating variations on poetic rhythms; the role of acoustics in harmonic consonance; the construction of musical scales, both familiar and exotic; mathematical models for rhythm and meter; the group theory of chords; geometrical models for counterpoint; and distance measures in music. Many of these topics are the subject of current research. Some are connected to open mathematical questions. Others have inspired composers to create new music.

Several books on mathematics and music aimed at a scientific audience have appeared recently. I mean to supplement these works, rather than repeat material available elsewhere. My criteria for the inclusion of a topic in this book are that it has validity within the field of music theory and is mathematically interesting.

Details

Intended audience

This book is written for a reader with a scientific background and some training in music—a reader who is comfortable with a degree of abstraction and who has an intellectual curiosity about mathematics and music. Although I assume familiarity with common mathematical notation, I will define terms that are specific to mathematics (that is, those that appear beyond the undergraduate calculus sequence). For example, I use variables and summation notation in Chapter 1 without introduction; however, in Chapter 5 I mean to define a mathematical group. Although calculus itself is not necessary to understand most of the material, the typical reader will have taken at least a year of calculus and have basic musical training, including the rudiments of music theory and notation. However, the background required varies from chapter to chapter. For example, Chapter 1 is accessible to a high school math student, and does not require any musical training.

Musicians, music theorists, and composers with some mathematical training are a secondary audience for the book. Moreover, although the book is not intended as a textbook, I use some of the topics described in the book in a course on mathematics and music that I teach.
Mathematical style

I believe that mathematics should be presented in the most direct way possible. Where there are choices of methods, I prefer those that do not require specialized knowledge. The techniques used in this book come from the fields of combinatorics, differential equations, number theory, discrete mathematics, group theory, geometry, and probability. I intend the book to be largely self-contained. I will explain, rather than invoke, results that are needed. For example, in Chapter 5 I will give a short proof of Burnside’s Lemma. Although, for the most part, I mean to avoid the straight “theorem-proof” style of mathematical exposition, I will state some theorems (Chapter 1 is atypical in this respect). I intend to provide proofs if they are relatively short and accessible. Where proofs are more technical, I will give the gist of the argument and refer the reader to the original source.

There are points of intersection between the topics in this book and more specialized mathematics. Some of my readers will be professional mathematicians who appreciate these connections. For example, tiling canons and the $\mathbb{Z}$-relationship discussed in Chapter 6 are related to the factorization of polynomials. I intend to mention advanced topics at the end of a section. Readers who are not familiar with the subject may skip to the next section.

What will the reader gain from this book?

For most readers, this will be a recreational mathematics book. However, the subject of math and music has surprising depth. I hope that the reader of this book will come to appreciate some of the multiple ways that mathematics is used to describe, analyze, and create music; be intrigued by the process of abstracting mathematical structure from music theoretical definitions; be inspired to make further investigations into the mathematics of music; and experiment with using mathematics to create music. Ultimately, my goal is that the reader gains a better understanding of both math and music.

Unique features

Several books on mathematics and music aimed at a scientific audience have appeared recently. However, this book fills a need not met by any other for two reasons.

First, the work of music theorists deserves to be better known to scientists. Many scientific writers have treated acoustics, sound recording, and sound synthesis; some have given cursory treatment to music theory (e.g. [5]). No writer has yet written in depth about
mathematical research done both by music theorists and mathematicians that is inspired by music theory. This field is growing quickly. Much of the research I describe has been done in the past ten years; some is connected to open mathematical problems.

Second, music theory is coming into its own as a field of applied mathematics and would benefit from a book that connects it to mainstream science. In the past, music theorists have worked on problems in isolation and, as a result, often reproved well-known mathematical results or labored too long over problems that were genuinely difficult. The fact that similar techniques arise in other applied mathematical fields (computational geometry, crystallography, physics, economics, and statistics) offers them a rich background on which to draw. In addition, several scientists (e.g. Sethares [52], Toussaint [15]) have done groundbreaking work in music that has escaped the attention of many music theorists.

Although I have classical training, my primary background is in traditional folk music, and this gives me a somewhat unusual perspective. I intend to draw upon examples from jazz, popular, and traditional music as well as from Western classical music.

What is not in this book?

I deliberately avoid several topics. This is not a book on music perception, performance analysis, or the styles of particular composers. I will give brief attention to acoustics, tuning systems, and sound synthesis and recording, as these are exhaustively covered by other authors (see Benson [5], for example).

My book is a comprehensive introduction to mathematical music theory as practiced in North America. The only significant topic in mathematical music theory that I intend to omit is musical topos theory, an application of category theory to music represented in the work of Mazzola [37] and chiefly developed and practiced in Europe. I do not believe the musical “payoff” to be large enough to warrant introduction of such heavy-duty machinery.

Competing books

Several books on mathematics and music have appeared recently:


Benson’s and Harkleroad’s books aim to be comprehensive introductions to mathematics and music. Their focus differs from mine in that they discuss acoustics, tuning systems, and electronic music more thoroughly than I intend, reference scientific and mathematical literature more than they reference the work of music theorists, have few references to popular or non-Western music, and almost completely ignore rhythm. The third and fourth books are collections of essays on mathematics and music by different authors; they do not aim to be comprehensive treatments of mathematics and music. Loy’s book is written for musicians rather than scientists.

Since my intention is to supplement these books, rather than to compete with them, there are only a few areas of overlap. Almost every book on math and music covers acoustics and tuning systems; my treatment will be relatively brief, as I primarily use these topics to motivate scale theory. Benson covers symmetry in music in one chapter (the remaining eight cover acoustics, tuning systems, and digital sampling and synthesis). This chapter overlaps with my discussion of transformations (Chapter 5); however, I intend to go into more depth. Chemillier’s article on African drumming [10] in the Springer book inspired Klingsberg’s and my research on asymmetric rhythms discussed in Chapter 6.

**Acknowledgements**

Much of the original research described in this book represents work that I have done with coauthors. By chapter, they are

Chapter 3: Krešimir Josić (section on the mathematics of musical instruments)

Chapter 6: Paul Klingsberg (section on asymmetric rhythms and tiling canons)

Chapter 6: Cliff Callender (section on the $Z$ relation)
Chapter 8: Dmitri Tymoczko

In addition, I am indebted to Manjul Bhargava for his help with Chapter 1.
Chapter-by-chapter summaries

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PREFACE

Summaries

PART I: INTRODUCTION

This brief introduction outlines some of the main themes of the book. I consider some commonalities of mathematics and music, including the role of abstraction and the fascination with symmetry and pattern. In addition, I give a short history of the relationship between the disciplines of mathematics and music theory from the ancient world to the present day.

PART II: MATHEMATICS AND MUSIC IN THE ANCIENT WORLD

1 Math for poets and drummers.

The systematic study of rhythm began in ancient India. Sanskrit poetry uses hundreds of meters, or poetic rhythms, each a pattern of short and long syllables. The scholar Pingala, who lived in the last few centuries B.C., found recursive algorithms that generate lists of meters. Curious patterns appearing in these lists led him to discover “Pascal’s” triangle—almost two millennia before the birth of Pascal—and the rudiments of the binary number system \[16, 64, 53\]. The problem of enumerating rhythms also inspired the medieval scholar Hemachandra to discover the “Fibonacci” numbers about fifty years before Fibonacci did. It is quite probable that Fibonacci learned the sequence through his contact with Indian mathematics.

Some of these early combinatorial problems apply to rhythm in the music of today. Many styles of dance music from around the world are identified with repeating rhythm patterns, the most well known of which is the clave from Cuban music. Hemachandra’s result may be generalized to count these patterns. In addition, a mnemonic used by Indian drummers to remember rhythm patterns is related to the theory of de Bruijn sequences \[55\].

1.1 Poetic meter in English and Sanskrit
1.2 Recursive and iterative algorithms for listing meters
1.3 Relationship to the binary numbers
1.4 Pingala’s meruprastāra and Pascal’s Triangle
1.5 The Hemachandra-Fibonacci numbers
1.6 Applications to musical rhythm
1.7 The Padovan numbers
1.8 A drummers’ mnemonic and its relationship to de Bruijn sequences
2 Acoustics and intervals.

The oldest identifiable musical instruments—bone flutes made tens of thousands of years ago—show that pitched sound has long been a central component of music. The ancient Greeks found that certain consonant intervals, or pleasing combinations of musical tones, are related to simple ratios of frequencies such as 2:1 (the octave) and 3:2 (the perfect fifth). This phenomenon had great mystical significance for them and led to an association between music and proportions that persists to the present day. Although many scientists, including Newton, Descartes, and Mersenne, experimented with sound, acoustic consonance was not fully explained until Helmholtz did so in the nineteenth century [20, 25].

This chapter introduces the principles of acoustics that form a foundation for the construction of scales and chords. The physical properties of musical instruments are closely correlated to the type of intervals perceived to be consonant when played on these instruments: the timbre (characteristic sound) of stringed and wind instruments favors intervals in common Western melodies, while bells, which have irrationally related harmonics, sound best using different intervals [52]. This phenomenon may explain the irrational intervals used in Indonesian gamelan music.

2.1 The physics of sound
2.2 The one-dimensional wave equation and its solutions
2.3 Instruments with rationally related harmonics
2.4 The wave equation in higher dimensions
2.5 Instruments with irrationally related harmonics
2.6 Musical intervals and consonance

PART III: ORGANIZING PITCH AND TIME

3 Scales.

Pitch is frequency measured on a logarithmic scale. A pitch class is a set of pitches that are equivalent modulo octave transposition: “middle C” is a pitch, while “C” is a pitch class. Pitch classes are often represented on the circle $\mathbb{R}/12\mathbb{Z}$, since an octave is twelve semitones.

A scale is an ascending or descending sequence of notes, or pitch classes, that impose structure on the pitch continuum. A scale determines a “unit step”; it essentially puts a metric on the pitch continuum (see [63]). The most common scales used in Western music are the five-note pentatonic scale, the seven-note diatonic scale, and
the twelve-note chromatic scale. Scales are often depicted as sets of points on the pitch class circle.

Although composers’ choice of scales is culturally dependent, there are quite a few principles of scale construction that are consistent across cultures. Scales need both harmonic and melodic potential: A scale should contain a good number of consonant intervals in order to permit pleasing harmonies, while its pitches should be spread approximately evenly on the pitch continuum in order to permit “smooth” melodies.

Since rational frequency ratios translate to irrational pitch intervals, compromises must be made in order for all the intervals in a scale to be more or less in tune. Tuning theorists use continued fraction approximations and other mathematical techniques to solve this problem. Scale and tuning theory are active areas of music theoretical research (see [5, 8, 24, 11, 61]).

3.1 Frequency, pitch, and pitch class
3.2 The diatonic and pentatonic scales
3.3 The theory of well-formed scales
3.4 Maximally even sets
3.5 Introduction to tuning theory
3.6 The Pythagorean scale and other generated scales
3.7 Continued fraction approximations

4 Meter and rhythm.

Musical events—changes in pitch, note attacks, harmonic shifts, even note releases—occur within a temporal framework called a meter, or measure of time. The variety and complexity of meters varies across cultures. Meter is often thought of as similar to the system of hours, minutes, and seconds: it determines a hierarchy among divisions of time [36]. However, meter subsumes a number of subtle stylistic phenomena that determine the “groove” or “feel” of a piece. In traditional dance music, the combination of meter with repetitive movement, both of dancers and musicians, suggests a more complex mathematical model involving coupled oscillators [31, 38, 58].

In contrast to a meter, a rhythm is a pattern of note onsets that is actually present in a piece. In traditional and popular music, repeated rhythmic patterns, or timelines, overlay the meter. Like a scale, a timeline is often represented on a circle. Meters and scales can have similar properties; for example, several common timelines are maximally even [59, 60].
4.1 Cross-cultural introduction to meter
4.2 Meter as division of time
4.3 Meter as movement
4.4 Microtiming and coupled oscillators
4.5 Rhythm timelines

PART IV: ALGEBRAIC METHODS

5 Transformations and generalized interval systems.

Transformational set theory—the theory of algebraic groups acting on scales, chords, melodies, and rhythms—is most often associated with atonal music [19, 57]. However, its roots are in the nineteenth- and twentieth-century combinatorial investigation of music [42], and theorists have recognized its usefulness in describing tonal music.

Transformational set theory represents chords or melodies as sets or sequences of pitches or pitch classes. The common musical transformations of transposition and inversion correspond to rotation and reflection of the pitch class circle, respectively. They do not commute. When restricted to twelve-tone equal temperament, these actions generate the dihedral group \( D_{24} \); in general, they generate the “infinite dihedral group” \( O(2) \).

Music theorists have typically classified chords by transposition class and set class: A chords’ transposition class is its equivalence class modulo transposition. We may think of a transposition class as a “chord type” such as a major triad. A chord’s set class is its equivalence class under both transposition and inversion. Minor and major triads form a set class. Mathematically, transposition classes and set classes correspond to binary necklaces and binary bracelets, respectively. These mathematical objects have a rich theory that also applies to chords [28].

Most musicians identify melodies or chord progressions that are related by transposition as more or less “the same.” Contextual transformations, or bijections between chords that preserve set class and commute with transposition, capture this similarity. The most famous of these are the neo-Riemannian transformations, which form a group of bijections between the set of major triads and the set of minor triads [48, 43, 12]. Some neo-Riemannian transformations commonly appear in music; examples include the “parallel” transformation that connects major and minor triads with the same root (C major and C minor) and the “relative” transformation that connects major and minor triads drawn from the same scale (C major and A minor). Representing contextual transformations as linear transformations on \( \mathbb{Z}_{12}^{n} \), \( \mathbb{Z}_{k}^{n} \), or even \((\mathbb{R}/12\mathbb{Z})^{n}\) introduces a more general theory (see [18, 21]).
Generalized interval systems, pioneered by Lewin [34, 35], are closely related to transformation groups. An “interval” in this context is a mapping from one chord or melodic motif to another; under certain conditions, intervals generate musically significant groups. Interval groups are required to act transitively; they are similar to what physicists call a torsor [32, 3].

5.1 Brief introduction to groups
5.2 Groups generated by transposition and inversion
5.3 Transposition classes and set classes
5.4 Enumeration using Burnside’s Lemma
5.5 Neo-Riemannian transformations
5.6 Contextual transformation groups
5.7 The representation theory of contextual transformations
5.8 Generalized interval systems
5.9 Torsors

6 Special sets: the Z-relation, asymmetric rhythms, and tiling canons.

This chapter concerns algebraic problems that arise in the study of certain chords and rhythms. Although the musical issues are specialized, they illustrate how seemingly simple musical problems involve nontrivial mathematics.

One invariant of transposition and inversion is the so-called interval vector—that is, the multiset of minimal distances between each pair of notes in the chord. Two chords with the same interval vector are called Z-related; a similar definition arises in crystallography, where sets with the same X-ray diffraction pattern are called homometric. The complete classification of homometric sets is an open question [50]; methods involve representing the problem as one of factoring certain polynomials. Although music theorists have explored the Z-relation extensively [54], they have hitherto been unaware of the work of crystallographers and mathematicians—work that raises serious issues about the status of the Z-relation in music.

Similar methods arise in a different musical problem. Rhythmic canons originate in the work of the twentieth-century composer Messiaen [40] and are related to so-called asymmetric rhythms that appear in African music [1, 10, 22]. The mathematician Vuza [65] showed that particular rhythmic canons correspond to sets forming a what mathematicians call a tiling of the integers. Although many mathematicians have studied this problem, the complete classification of integer tilings—and hence tiling canons—is an open question [14, 22]. As with the phase retrieval problem, the existence of integer tilings is related to the factorization of polynomials.
PART V: THE GEOMETRY OF MUSIC

7 Geometrical music theory.

Music theorists have frequently invoked geometry in modeling musical objects such as chords, rhythms, and scales; however, no unified geometric perspective has hitherto emerged. In their recent articles, Callender, Quinn, and Tymoczko (CQT) [6, 62, 7] demonstrate that many musical terms can be understood as expressing symmetries of \( n \)-dimensional space. Identifying—"gluing together"—points related by these symmetries produces a family of non-Euclidean quotient spaces that subsume a large number of geometric models proposed in the literature. These models are the first to accommodate both transformational set theory and a topological viewpoint that holds that two chords are "close" if they can be related by minimal changes in their notes.

Geometrical music theory provides a framework within which to study the connections between chords or set classes. Points that are close to each other in a quotient space represent chords that may be connected by voice leadings (mappings between chords) in which each voice moves by a short distance. Chords employed in Western common practice music typically lie near an equal division of the octave, thus permitting efficient voice leadings between them.

The algebraic relationships introduced in Chapter 5 can be realized as group actions on these spaces in two ways: one involves the linear transformations developed in Chapter 5 and the other involves a "bundle" construction [21].

7.1 The OPTIC equivalence relations
7.2 Musical quotient spaces (orbifolds)
7.3 Individual and uniform equivalence of progressions
7.4 Voice leadings as paths in quotient spaces
7.5 Contextual transformations as group actions on CQT spaces
7.6 Fiber bundle structure

8 On the measurement of music.

Chord similarity and set class similarity have occupied music theorists for the last century [47, 33, 19, 46, 41]. Representing musical structures geometrically presents the possibility of modeling certain musical similarities by distance functions. In this chapter, we consider distance measures on voice leadings and rhythms.

In general, a method of measuring voice leadings is a way of comparing distances in CQT spaces. Although these spaces locally resemble $\mathbb{R}^n$, we are not forced to use the quotient metric inherited from Euclidean distance. In fact, music theorists have proposed a number of ways of measuring voice leadings; the challenge is to define “distances” in CQT spaces in a manner flexible enough to accommodate a number of voice-leading measures. Tymoczko [62] proposed a “distribution constraint” that any reasonable method of comparing voice leadings should satisfy. The distribution constraint imposes the submajorization partial order, familiar from applied mathematics, on distances in CQT spaces [23]. Identical mathematical techniques arise in welfare economics and statistics.

Toussaint [60, 15] pioneered the use of distance measures to perform phylogenetic analysis on rhythm timelines in flamenco and African music. He was able to reconstruct an “ancestral” flamenco rhythm by studying distances between flamenco rhythms used today.

8.1 The distribution constraint
8.2 Submajorization and convex functions
8.3 Application to distance between set classes
8.4 Distance functions in rhythmic analysis
8.5 Phylogenetic analysis of rhythm timelines

PART VI: MATH AND MUSIC IN PRACTICE

9 Mathematics and composition.

Many composers and mathematicians, going back at least to Pingala, have experimented with using mathematics to explore musical possibilities. The extent to which composers explicitly use math has varied throughout history. Seventeenth century change-ringing patterns and eighteenth century dice games are the first examples of mathematically generated compositions [17, 2]. Serious explorations of
mathematics in music appear in the work of twentieth-century composers such as Xenakis, Messiaen, and Schoenberg [66, 49, 44, 40, 27]. Their mathematical influences include combinatorics, probability, and chaos theory.

9.1 Using group theory to generate variations: from change-ringing to “Clapping music”

9.2 The mathematics of Bach’s Goldberg Variations

9.3 Serialism and combinatorics

9.4 Introduction to probability

9.5 The music of chance: dice games, Markov chains, and fractal music

**AUDIO EXAMPLES** Audio examples for the book will be available as mp3s on my personal web site [http://www.sju.edu/~rhall](http://www.sju.edu/~rhall). Sample audio examples of asymmetric rhythms and tiling canons (Chapter 6) are available at [http://www.sju.edu/~rhall/Rhythms/AsymmetricRhythms/](http://www.sju.edu/~rhall/Rhythms/AsymmetricRhythms/).

**FIGURES** Except where indicated, I created the figures in this proposal. Although color is ideal, all these figures could be reproduced in black and white. Even a few color plates would help illustrate the geometrical models in Chapter 7.


About the author

I have an unusual background: although I am a career mathematician, I majored in Ancient Greek as an undergraduate, studied traditional music in Scandinavia and the Shetland Islands on a Watson Fellowship, and continue to perform and record as a folk musician. I have never been comfortable labeling myself as purely a scientist; rather, I have pursued research that crosses into the humanities and have sought out interdisciplinary collaboration. I firmly believe that, rather than being a distraction, the fact that I move between several roles gives me a richer perspective on each.

Although I have classical training, I grew up playing traditional folk music, primarily of the British Isles and Anglo-America. My current love is Scandinavian music. I play English concertina, piano, and tabla (Indian drums).

As a musician, I understand the importance of promoting one’s own work. I intend to promote this book actively. I especially enjoy giving talks and am quite willing to travel. Since 2004, I have given invited addresses at the Bridges conference in Spain and at my regional MAA meeting; eleven presentations at international, national, and regional meetings, including both the national math meetings and the national music theory meeting; ten colloquium talks, including one at IRCAM in Paris; and two invited workshops for math faculty on incorporating Ethnomathematics, including the math of music, into the mathematics classroom. In addition, I am well connected in the Math and the Arts community.

Biography

Here is my “official” bio:

RACHEL WELLS HALL received a B.A. in Ancient Greek from Haverford College and a Ph.D in mathematics from the Pennsylvania State University in the field of operator algebras. Her research interests include applications of mathematics to music and Ethnomathematics. She is on the editorial boards of Music Theory Spectrum, the Journal of Mathematics and Music, and the Journal of Mathematics and the Arts. As a member of the folk music trio Simple Gifts since 1995, she has toured throughout the Mid-Atlantic and released three albums. She plays the concertina, piano, and tabla. See the attached CV for more details.
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Introduction

Note: the figures that accompany this chapter should be interspersed with the text in the final manuscript, as they are an essential component of this introduction.

The idea that mathematics and music are intimately connected has been widely accepted for centuries, both in the West and the East. Music was one of the first subjects of applied mathematics. Historically, there have been two streams of mathematical investigation of music: the study of the properties of sound and the study of musical patterns. Both of these topics influenced the development of mathematics.

Interest in the physics of music has undoubtedly played a part in the history of science and mathematics. The Pythagoreans’ discovery of the mathematical nature of musical sound gave rise to an association between music and proportions that persists in Western thought to the present day. Acoustics motivated experimental scientists from Pythagoras to Descartes, Euler, Hooke, Newton, and Helmholtz. Although some early investigations were more mystical than scientific, quite a few, including the discovery of sound waves, had real import. The heirs to this tradition are the engineers and computer scientists who pioneered sound recording and sound synthesis.

Interest in music inspired Helmholtz’s experiments with sound. Here, a needle attached to a tuning fork sketches a sinusoidal curve [25, p. 20].
The second direction of exploration concerns pattern. Questions about variation, similarity, enumeration, and classification of musical structures are essentially mathematical ones, and, in fact, musical problems inspired a great deal of combinatorial investigation. In ancient India, exploration of rhythmic patterns led to the discovery of the Fibonacci numbers and Pascal’s triangle. Attempts to find algorithms—sets of mathematical instructions—that generate musical possibilities first surfaced in the work of Sanskrit prosodists and intrigued the mathematicians Mersenne, Leibniz, Bernoulli, and Euler, among others. Although modern-day music theorists and composers of algorithmic music continue in this tradition, the disciplines of mathematics and music theory have been largely separate since the eighteenth century.

Music was central to early combinatorics. Mersenne’s *Harmonie Universelle* contains solutions to many combinatorial problems that arise in music, including this table of the twenty-four melodies made from the notes mi, fa, sol, and la [39]. He also recognized that binomial coefficients—entries in Pascal’s triangle—count melodies where a certain number of repeats are permissible.

However, it is surprisingly difficult to explain our continuing fascination with the connection between music and math. Indeed, the presence of mathematical structure in music is hardly remarkable. It is said that mathematics is the science of pattern; patterns abound in the natural and man-made worlds; therefore, we expect to find mathematics in music, just as we have found mathematics in physics, biology, economics, and many other fields. In fact, the cliché “math and music go together” is memorable for its seeming contradiction: how can a subject that appears to many people soulless and abstract apply to something as emotional and accessible as music? Although math has inspired some composers, music determined by rigid mathematical rules is rarely successful. And, in mathematics, truth, not beauty, is the ultimate arbiter; a difficult proof is nonetheless a proof.
Despite this, I remain convinced that there is a deeper connection between math and music—a connection that goes far beyond the importance of proportions in constructing scales and chords, or the usefulness of combinatorics in generating lists of patterns.

Unlike visual art, but like pure mathematics, most music is abstract—it does not represent anything in the physical world. Both composers and mathematicians grapple with the difficulties of describing abstract forms. Readers of mathematics, like performing musicians, must recreate these structures for themselves. Precise notation is crucial. However, notation limits the possibilities we can represent or comprehend; quite a few composers have experimented with nonstandard notation.

Although composers differ in the extent to which they explicitly use mathematics, they all must organize sound in several dimensions at once: time, pitch, volume, timbre, and so on. Comprehending a piece thus requires the ability to make connections on different levels simultaneously—a skill that mathematics also requires. Even one of our first mathematical discoveries, the positional number system, relies on recognizing nested structures.

Perhaps because of these challenges, music is more explicitly patterned than the other arts. Composers often employ repetition, variation, similarities of scale, and other symmetries. Music theorists have increasingly looked to mathematics for the vocabulary to describe these relationships.

A few years ago, I heard a group of music theorists present their research at a national math conference. They described their work in much the same manner as mathematicians do—that is, in the language of theorem and proof. Although I had begun doing research on the applications of mathematics to music, the topics with which I was familiar—acoustics, tuning systems, digital synthesis—were quite different from the discrete mathematics, geometry, and algebra these music theorists used to model higher cognitive structures such as melody, harmony, scales, and rhythm. Curiosity about their work eventually led me to take new directions in my own research. I discovered that music theorists have considered problems that are appealing, nontrivial, and, in some cases, connected to deep mathematical questions.

With this book, I aim to present the work of music theorists in a manner that is accessible to a scientist with some musical training and to explain how problems in music theory connect to problems that arise in mathematics. I hope that the reader will come to appreciate some of the multiple ways that mathematics is used to describe music, be intrigued by the process of abstracting mathematical structure from music theoretical definitions, be inspired to make further investigations into the mathematics of music, and experiment with using mathematics to create new music.
Both composers and mathematicians grapple with the difficulties of notation. The earliest extant music notation is a Babylonian tablet from 2000–1700 BC that contains an array of cuneiform numbers defining two heptatonic scales [13]. An eighteenth-century Tibetan manuscript with graphic music notation creates an interesting juxtaposition with a fragment of Sylvano Bussotti’s score for his *Rara Requiem* (1969) [13].
Several composers have explicitly used mathematics in their work. Schoenberg constructed this “slide rule” and tone row table to write his *Suite for three clarinets, violin, viola, cello, and piano, op. 29* (1925–6) [9].

Music is more explicitly patterned than the other arts. A graphical depiction of four of Bach’s *Canons on the Goldberg Ground* shows his use of the mathematical operations of reflection and glide reflection.
These models for chord relationships are examples of orbifolds, or singular quotients of $\mathbb{R}^n$. Geometrical music theory was developed in a series of papers by Callender, Quinn, and Tymoczko [6, 62, 7] (2004–2008).
Chapter 1

Math for Poets and Drummers

... But most by Numbers judge a Poet’s Song,
And smooth or rough, with them, is right or wrong;
...
These Equal Syllables alone require,
Tho’ oft the Ear the open Vowels tire,
While Expletives their feeble Aid do join,
And ten low Words oft creep in one dull Line,
...
—Alexander Pope, An Essay on Criticism (1709)

According to Pope, math and poetry don’t mix. Critics who judge poetry solely on how well it conforms to rhythmic rules are missing the point—worse, they encourage dull, repetitive poetry. Mathematicians, however, have found intriguing patterns in these rhythms. This chapter tells the story of the discovery of Pascal’s Triangle and the Fibonacci numbers, not by Pascal and Fibonacci, but by scholars in ancient India who were interested in the rhythms of poetry.

In English, a poetic rhythm, called a meter, is a pattern of stressed and unstressed syllables, indicated by the symbols — and ⌢, respectively. English poets use about a dozen different meters. Iambic pentameter—five pairs of alternating unstressed and stressed syllables to a line—is the meter of Shakespeare’s plays and numerous poems.¹

¹The delightful iambic pentameter quadratic \(x^2 + 7x + 53\) appears in Four Riddles by the mathematician and writer Charles Dodgson, also known as Lewis Carroll.
including Pope’s *An Essay on Criticism*. Pope parodies the sing-song rhythm produced when poets use iambic pentameter too strictly:

\[ \text{And ten low Words oft creep in one dull Line} \]

In Sanskrit, the classical language of India, meters are based on duration, rather than accent, with a long syllable (*guru*) having twice the length of a short one (*laghu*). Sanskrit meters fall into two categories: meters in which the number of syllables in a line are fixed, and meters in which the duration of a line is fixed, but not the number of syllables. While English poets use relatively few meters, there are *hundreds* of Sanskrit meters; this may explain why ancient Indian mathematicians found meter so fascinating.

**Pingala and the binary representation of meter**

Pingala was the first of a long line of Indian scholars who studied meter mathematically. He probably lived in the last few centuries BC.\(^2\) Pingala’s writings took the form of short, cryptic verses, or *sūtras*, which served as memory aids for a larger set of concepts passed on orally. We know Pingala’s work chiefly from medieval commentators, including Halāyudha (13th century). Kedāra Bhatt (8th century) solved the same problems that Pingala did; however, his mathematical methods are strikingly different from Pingala’s [64, 30].

One of Pingala’s most impressive achievements was developing an algorithm—a finite set of instructions—that generates a catalog in which each pattern of long and short syllables appears exactly once. Pingala provided a key for finding the position of any given pattern in this catalog and even a way to reconstruct a lost portion of the catalog.

Pingala’s *sūtras* address four problems, here expressed in modern-day notation:

**Problem 1.** How can we systematically list all the patterns of *n* syllables for any *n*?

**Problem 2.** Suppose a pattern were erased from this list. How can we reconstruct the lost pattern?

**Problem 3.** Given any pattern, how can we find its position on the list without recreating the entire list?

\(^2\)Some modern scholars think he lived around 500 BC and was the nephew of the grammarian Panini; others claim he lived around 200 BC. However, the earliest definitive reference to his writing comes in the third century AD [64].
Problem 4. What is the total number of patterns of $n$ syllables?

Problem 1: listing the patterns of $n$ syllables.

Pingala gave instructions on how to list the patterns of $n$ syllables in a table he called prastāra, or expansion. The prastāra of one syllable has two elements (a long syllable and a short syllable, represented by the Sanskrit symbols $\bar{s}$ and $\bar{l}$, respectively). We are instructed to “mix” the one-syllable prastāra with itself to get the two-syllable prastāra. That is, “mix” $\bar{s}$ and $\bar{l}$ with $\bar{s}$ to get $\bar{s}\bar{s}$ and $\bar{s}\bar{l}$; mix $\bar{s}$ and $\bar{l}$ with $\bar{l}$ to get $\bar{s}\bar{l}$ and $\bar{l}\bar{l}$. To get the three-syllable prastāra, append $\bar{s}$ to the end of the two-syllable prastāra, then do the same for $\bar{l}$. These instructions generalize to any length of pattern. Prastāras of one through four syllables are shown in Figure 1.1.

Kedāra Bhatt found a different algorithm that lists the $n$-syllable patterns in exactly the same order Pingala did [64, 30]. The first pattern on the list consists of $n$ long syllables. Suppose you are given any pattern on the list (for example, $\bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{s}$). To get the next pattern, start from the left by writing long syllables. When you reach the position of the first long syllable in the previous pattern, write a short syllable:

$$
\begin{array}{cccccccc}
1 & \bar{s} & 1 & \bar{s}\bar{s} & 1 & \bar{s}\bar{s}\bar{s} & 1 & \bar{s}\bar{s}\bar{s}\bar{s} \\
2 & \bar{l} & 2 & \bar{s}\bar{l} & 2 & \bar{s}\bar{s}\bar{l} & 2 & \bar{s}\bar{s}\bar{s}\bar{l} \\
3 & \bar{s}\bar{l} & 3 & \bar{s}\bar{s}\bar{l} & 3 & \bar{s}\bar{s}\bar{s}\bar{l} & 3 & \bar{s}\bar{s}\bar{s}\bar{s}\bar{l} \\
4 & \bar{l}\bar{l} & 4 & \bar{s}\bar{l}\bar{l} & 4 & \bar{s}\bar{s}\bar{l}\bar{l} & 4 & \bar{s}\bar{s}\bar{s}\bar{l}\bar{l} \\
5 & \bar{s}\bar{s}\bar{l} & 5 & \bar{s}\bar{s}\bar{s}\bar{l} & 5 & \bar{s}\bar{s}\bar{s}\bar{s}\bar{l} & 5 & \bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{l} \\
6 & \bar{s}\bar{s}\bar{s}\bar{l} & 6 & \bar{s}\bar{s}\bar{s}\bar{s}\bar{l} & 6 & \bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{l} & 6 & \bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{l} \\
7 & \bar{s}\bar{s}\bar{s}\bar{s}\bar{l} & 7 & \bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{l} & 7 & \bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{l} & 7 & \bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{l} \\
8 & \bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{l} & 8 & \bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{l} & 8 & \bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{l} & 8 & \bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{l} \\
\end{array}
$$

Figure 1.1: Prastāras of one-, two-, three-, and four-syllable meters, with indices

Then recopy the rest of the previous pattern:

$$
\begin{array}{cccccccc}
1 & \bar{s} & 1 & \bar{s}\bar{s} & 1 & \bar{s}\bar{s}\bar{s} & 1 & \bar{s}\bar{s}\bar{s}\bar{s} \\
\bar{s} & \bar{s} & 1 & \bar{s}\bar{l} & 1 & \bar{s}\bar{s}\bar{l} & 1 & \bar{s}\bar{s}\bar{s}\bar{l} \\
\bar{s} & \bar{s} & \bar{s} & \bar{l}\bar{l} & \bar{s} & \bar{s}\bar{s}\bar{l} & \bar{s} & \bar{s}\bar{s}\bar{s}\bar{l} \\
\bar{s} & \bar{s} & \bar{s} & \bar{s}\bar{l} & \bar{s} & \bar{s}\bar{s}\bar{l} & \bar{s} & \bar{s}\bar{s}\bar{s}\bar{l} \\
\bar{s} & \bar{s} & \bar{s} & \bar{s}\bar{s}\bar{l} & \bar{s} & \bar{s}\bar{s}\bar{s}\bar{l} & \bar{s} & \bar{s}\bar{s}\bar{s}\bar{s}\bar{l} \\
\bar{s} & \bar{s} & \bar{s} & \bar{s}\bar{s}\bar{s}\bar{l} & \bar{s} & \bar{s}\bar{s}\bar{s}\bar{s}\bar{l} & \bar{s} & \bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{l} \\
\bar{s} & \bar{s} & \bar{s} & \bar{s}\bar{s}\bar{s}\bar{s}\bar{l} & \bar{s} & \bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{l} & \bar{s} & \bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{l} \\
\bar{s} & \bar{s} & \bar{s} & \bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{l} & \bar{s} & \bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{l} & \bar{s} & \bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{s}\bar{l} \\
\end{array}
$$
Pingala’s algorithm follows naturally from the observation that the list of \((n - 1)\)-syllable patterns is twice nested within the list of \(n\)-syllable patterns. Bhatt’s algorithm is less obvious; it is perhaps easiest to derive from the routine described in his solution to Problem 3 (discussed below).

The computer scientist Donald E. Knuth credits Sanskrit scholars with “the first-ever explicit algorithm for combinatorial sequence generation” \([29]\), meaning they were the first to develop a process for systematically listing patterns with a given property. Bhatt’s algorithm is \textit{iterative}—that is, it gives instructions to get from one pattern on the list to the next—while Pingala’s algorithm is \textit{recursive}—it generates the entire list of \(n\)-syllable patterns from the list of \((n - 1)\)-syllable patterns, so that the list of patterns of any given length may be generated by repeatedly invoking the same routine. Both iteration and recursion are fundamental computer programming techniques. A fascination with recursion appears in Indian art and religion from ancient times. For example, the medieval Kandariya Mahadeva temple (Figure 1.2) contains several miniature copies of itself.

\textbf{Problem 2: recovering a lost pattern.}

Suppose a pattern were erased from the list. How can we recover it without having to regenerate the entire list?

Pingala’s pattern recovery algorithm assumes that we know the position of the missing pattern, which we will call its \textit{index} (see Figure 1.1). He gives the following instructions: if the index can be halved, halve it and write \(\mathcal{S}\); otherwise, write \(1\), add one, and halve the result. Repeat the process, writing from left to right, until the pattern has the correct number of syllables.

To understand why Pingala’s algorithm works, let \(w\) represent a string of \(n\) characters drawn from the set \(\{\mathcal{S}, 1\}\) and let \(\text{ind} \ w\) denote its index, which is odd if \(w\) starts with \(\mathcal{S}\) and even if \(w\) starts with \(1\). If we remove the first syllable of pattern \(w\) to get a new pattern, \(w'\), then

\[
\text{ind} \ w' = \begin{cases} 
  \frac{(\text{ind} \ w + 1)}{2} & \text{if \ \text{ind} \ w \ \text{is odd}} \\
  \frac{(\text{ind} \ w)}{2} & \text{if \ \text{ind} \ w \ \text{is even}} 
\end{cases}
\]

Suppose we know the index of pattern \(p\), but not the pattern itself. Since we use a repeated routine, it is convenient to rename \(p\) as \(p_n\). We recover the syllables of \(p_n\) one at a time; at each point the string of unknown syllables is one shorter, and we call these successively shorter strings \(p_{n-1}, p_{n-2}, \ldots, p_1\). If \(\text{ind} \ p_n\) is odd, \(p_n\) begins with \(\mathcal{S}\); therefore, \(p_n = \mathcal{S}p_{n-1}\), where \(\text{ind} \ p_{n-1} = (\text{ind} \ p_n + 1)/2\). If \(\text{ind} \ p_n\) is even, \(p_n\) begins with \(1\); therefore, \(p_n = 1p_{n-1}\), where \(\text{ind} \ p_{n-1} = \text{ind} \ p_n/2\). We now know the index of \(p_{n-1}\) in the list of patterns of length
Figure 1.2: Recursion in Indian architecture: the Kandariya Mahadeva temple [45]
$n - 1$. All the characters of $p_n$ can be generated by repeating this algorithm.

The following steps show that the fifth pattern of five syllables is $\textit{SSISS}$:

<table>
<thead>
<tr>
<th>pattern</th>
<th>$\text{ind } p_i$</th>
<th>parity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_5$</td>
<td>5</td>
<td>odd</td>
</tr>
<tr>
<td>$\textit{S}p_4$</td>
<td>$(5 + 1)/2 = 3$</td>
<td>odd</td>
</tr>
<tr>
<td>$\textit{SS}p_3$</td>
<td>$(3 + 1)/2 = 2$</td>
<td>even</td>
</tr>
<tr>
<td>$\textit{SSI}p_2$</td>
<td>$2/2 = 1$</td>
<td>odd</td>
</tr>
<tr>
<td>$\textit{SSIS}p_1$</td>
<td>$(1 + 1)/2 = 1$</td>
<td>odd</td>
</tr>
<tr>
<td>$\textit{SSISS}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Bhatt does not address this problem, as his algorithm generates the missing row from the previous pattern.

**Problem 3: finding the index of a pattern.**

Suppose we are given a pattern. Where does it belong on the list?

Pingala’s indexing process reverses the algorithm he developed for Problem 2. The index of the pattern of all long syllables is one. For any other pattern, start with the first short syllable from the right. The instruction is simply “multiply by two”; we assume that the starting number is one. If the next syllable on the left is $\textit{S}$, again multiply the resulting number by two; otherwise, multiply it by two and subtract one. Repeat this process until the leftmost character is reached.

This procedure stems from the observation that if $w$ is a pattern,

\[
\text{ind}\textit{S}w = 2\text{ind } w - 1 \\
\text{ind}\textit{I}w = 2\text{ind } w
\]

Since adding any number of $\textit{S}$’s to the end of a word does not change its index, Pingala’s process begins with the first short syllable from the right. While the original pattern is recreated by adding one syllable at a time on the left, the algorithm keeps track of the index of the current pattern.

The following steps show that the index of $\textit{SSISS}$ is five.
Bhatt’s algorithm for finding the index of a given pattern is again strikingly different from Pingala’s. He essentially assigned a place value to each syllable. Reading from the left, the first place has value one, the second has value two, the third has value four, and so on, so that the value of the $i$th place is $2^{i-1}$.

Bhatt observed that the index of a pattern is one more than the sum of the place values of its short syllables. For example, the index of $\text{ISSI}$ is six, because short syllables fall in the first and third columns:

\[
1 + 1 + 0 \cdot 2 + 4 + 0 \cdot 8 = 6
\]

The reason Bhatt’s algorithm works is that if $w$ is a meter of $n$ syllables,

\[
\text{ind } wS = \text{ind } w
\]

\[
\text{ind } wI = \text{ind } w + 2^n.
\]

Perhaps Bhatt was predisposed to use a positional indexing system, as the positional decimal numbers are thought to have been adopted in India around the time of his birth [26].

We may now return to Bhatt’s somewhat opaque solution to Problem 1. Suppose we know the $k$th pattern of $n$ syllables, and wish to write the $(k+1)$st pattern. We define $k_1, k_2, \ldots, k_n$ by

\[
k_i = \begin{cases} 
0 & \text{if the } i\text{th syllable is } S \\
1 & \text{if the } i\text{th syllable is } I 
\end{cases}
\]

Then

\[
k = 1 + k_1 + 2k_2 + 4k_3 + \ldots + 2^{n-1}k_n.
\]

Adding one to both sides of the equation has a “rollover effect”: if the list $k_1, k_2, \ldots, k_n$ begins with a string of 1s, to get the $(k+1)$st pattern, change all of these to 0s, replace the first 0 with 1, and leave the rest of the sequence alone.
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Problem 4: counting patterns of \( n \) syllables.

The fourth problem Pingala and Bhatt tackled involves counting the possible poetic meters of a fixed number of syllables [4]—in other words, evaluating \( 2^n \).

Bhatt gives two algorithms [30]. The first is based on his solution to the index-finding problem. He observes that the index of the last pattern (\( n \) short syllables) is \( 2^n \). Using his algorithm, this equals one plus the sum of the positional values of the syllables; in other words,

\[
2^n = 1 + \sum_{i=1}^{n} 2^{i-1}.
\]

Bhatt’s second algorithm involves summing the binomial coefficients \( nC_r \); we will discuss these in the next section.

Pingala, on the other hand, gives a recursive algorithm based on the observation that

\[
2^n = \begin{cases} 
(2^{n/2})^2 & \text{(used if } n \text{ is even)} \\
2^{n-1} \cdot 2 & \text{(used if } n \text{ is odd)}
\end{cases}
\]

(although these statements are equivalent, Pingala would not have been able to evaluate \( 2^{n/2} \) for \( n \) odd) [30]. For example, we calculate \( 2^9 \):

\[
2^9 = 2^8 \cdot 2 = (2^4)^2 \cdot 2 = ((2^2)^2)^2 \cdot 2.
\]

The binary number system

In some ways, Pingala and Bhatt anticipated the development of the binary number system, which was not fully described until Gottfried Leibniz did so in the seventeenth century.\(^3\) The normal decimal-to-binary conversion procedure is quite similar to Pingala’s process for finding an unknown pattern given its index. In this case, the decimal number serves as the “index.” To find a binary number if its decimal value is known, divide the decimal by two, write the remainder, and continue this process, writing the successive remainders on the left.

In contrast, Bhatt’s algorithm, which assigns a positional value to each syllable, recalls the

\(^3\)The binary number system is a base-two positional number system. It has two digits, 0 and 1, and its place values are powers of two. Thus, the decimal numbers 1, 2, 8, and 11 have binary representation 1, 10, 1000, and 1011, respectively.
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binary-to-decimal conversion formula

\[ b_n b_{n-1} \ldots b_1 b_0 \text{ (base 2)} = \sum_{i=1}^{n} b_i 2^i \]

where \( b_i \in \{0, 1\} \) are the digits of a binary number.

Despite these similarities, it is difficult to credit Pingala or Bhatt with the discovery of the binary numbers. Neither author presented his indexing procedure as a number system. Their systems were not used to perform computations, or indeed to index anything other than poetic meters. The convention of assigning the index one, rather than zero, to the first pattern makes computations problematic (giving \( SS \ldots SS \) the index zero was not an option at the time—in fact, the Indians didn’t consider zero a number until about the fifth century AD). Moreover, the correspondence between metrical patterns and their indices is one-to-one only if the number of syllables is fixed; this is like considering 1, 01, and 001 to be distinct numbers.

Pingala also developed a different way of cataloging meters that was more prevalent—in fact, it is still used by poets and drummers today. This is considered in the last section of this chapter.

Pascal’s Triangle and the Expanding Mountain of Jewels

Pingala is also credited with the discovery of “Pascal’s” Triangle in India, which he called the meruprastāra, or “the expanding mountain of jewels” (meru is a mythical mountain made of gold and precious stones, and prastāra is the word for expansion). This triangle describes the number of combinations of \( n \) syllables, taken one at a time, taken two at a time, and so on, where each syllable is considered different, rather than just long or short. When each list counting combinations of \( r \) syllables drawn from sets of \( n \) syllables is arranged horizontally, and successive lists are stacked, the numbers form a triangular array that extends indefinitely:

\[
\begin{array}{cccccc}
  & 1 &   &   &   & \\
1 & 1 &   &   &   & \\
1 & 2 & 1 &   &   & \\
1 & 3 & 3 & 1 &   & \\
1 & 4 & 6 & 4 & 1 & \\
\end{array}
\]
Pingala recognized that each interior number is the sum of the two numbers above it. This array is known as Pascal’s triangle—though, of course, it wasn’t yet named for Pascal, who was born in France in 1623.

The roof of the Kandariya Mahadeva temple (Figure 1.2) depicts Mount Meru surrounded by smaller copies of itself. Note that the meruprastāra is also made up of smaller copies of itself—just add the corresponding entries in the smaller triangles:

\[
\begin{array}{ccccccc}
1 & 1 & - & & & & \\
1 & 2 & 1 & - & & & \\
1 & 3 & 3 & 1 & - & & \\
1 & 3 & 3 & 1 & - & & \\
\end{array} + \begin{array}{ccccccc}
1 & 1 & - & & & & \\
1 & 2 & 1 & - & & & \\
1 & 3 & 3 & 1 & - & & \\
1 & 3 & 3 & 1 & - & & \\
\end{array} = \begin{array}{ccccccc}
1 & 1 & & & & & \\
1 & 2 & 1 & - & & & \\
1 & 3 & 3 & 1 & - & & \\
1 & 3 & 3 & 1 & - & & \\
\end{array}
\]

Bhatt discovered the same triangle, but in a different context: he found the number of meters of \( n \) syllables having \( r \) short syllables. The fact that the two problems have the same solution is, of course, no coincidence. The number of ways of choosing \( r \) syllables from a collection of \( n \) different syllables (\( \binom{n}{r} \) in modern notation) equals the number of ways of choosing the \( r \) locations for the short syllables within a meter of \( n \) short or long syllables. For example, the choice of \{2, 5, 7\} from the collection \{1, 2, \ldots, 8\} corresponds to the eight-syllable meter that has short syllables in positions 2, 5, and 7 (\textit{SISSISIS}.

Returning to Problem 4, Bhatt’s interpretation of the triangle shows that the sum of the entries in the \( n \)th row of the triangle gives the total number of meters of \( n \) syllables. In modern notation, Bhatt’s second solution to Problem 4 is the formula

\[2^n = \sum_{r=0}^{n} \binom{n}{r}.
\]

Neither of the Indian authors explains the relationship between the addition rule for obtaining successive rows of the meruprastāra and the structure of the meters they represent. However, given Pingala’s fondness for recursive rules, he may have observed that there is a one-to-one correspondence between the ways of choosing \( r \) objects from a collection of \( n \) objects and the ways of choosing either \( r - 1 \) objects out of \( n - 1 \) objects or \( r \) objects out of \( n - 1 \) objects. For example, there are ten three-element combinations of \{1, 2, 3, 4, 5\}. Partition these into combinations that contain 5 and combinations that do not. If a combination is to contain the element 5, choose the other two elements from the set \{1, 2, 3, 4\} in \( \binom{4}{2} \) ways; add to this the \( \binom{4}{3} \) three-element combinations that do not contain 5 to arrive at the equation \( \binom{5}{3} = \binom{4}{2} + \binom{4}{3} \). Lists of meters combine in the same way: in this case, partition the five-syllable meters into those with the fifth syllable short
The 12th-century writer Acarya Hemachandra also studied poetic meter [53]. Instead of counting meters with a fixed number of syllables, Hemachandra counted meters having a fixed duration, counting short syllables as one beat and long syllables as two beats, as shown in Figure 1.3. The numbers of patterns form the sequence 1, 2, 3, 5, 8, \ldots. Hemachandra discovered that each entry is found by adding the two previous. In other words, he found the “Fibonacci” numbers—half a century before Fibonacci! Indian poets and drummers know these numbers as “Hemachandra numbers.”

Hemachandra not only commented that the Hemachandra-Fibonacci numbers count meters, but also explained why they do so. Partition the collection of patterns of duration

and those with the fifth syllable long.\footnote{The 12th-century writer Bhaskara gives yet another algorithm in his \textit{Lilavati} [16]. To find the \textit{n}th row in the meruprastāra, start by writing the numbers 1, 2, \ldots, \textit{n}, and above them write the numbers \textit{n}, \textit{n}−1, \ldots, 2, 1, like so (shown for \textit{n} = 5):}

\begin{center}
\begin{tabular}{cccccc}
1 & 2 & 3 & 4 & 5 \\
\hline
1 & 2 & 3 & 4 & 5
\end{tabular}
\end{center}

The first number in the row is 1 (this is true for every \textit{n}). Obtain the other numbers in the row by successively multiplying and dividing by the numbers you have written:

\begin{align*}
1 \cdot 5/1 &= 5; & 5 \cdot 4/2 &= 10; & 10 \cdot 3/3 &= 10; & 10 \cdot 2/4 &= 5; & 5 \cdot 1/5 &= 1.
\end{align*}

This algorithm is iterative; you do not have to generate any previous rows in order to find row \textit{n}. Although he does not make a connection to the recursive addition rule found by Pingala, Bhaskara comments that the \textit{n}th row of the meruprastāra counts both the number of ways of choosing \textit{r} of \textit{n} different objects and the number of ways of arranging \textit{r} objects of one kind and \textit{n}−\textit{r} of another. He also notes that poetic meter is only one of the possible applications of the meruprastāra.
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$n + 1$ into two disjoint sets: patterns of duration $n$ followed by a short syllable and patterns of duration $n - 1$ followed by a long syllable. Let $H_n$ represent the total number of patterns of duration $n$. The partition shows that $H_{n+1} = H_n + H_{n-1}$. Since $H_1 = 1$ and $H_2 = 2$, we obtain the Hemachandra-Fibonacci sequence.

Hemachandra’s problem is equivalent to the “domino-square problem”: in how many ways can $1 \times 2$ dominoes and $1 \times 1$ squares tile a $1 \times n$ rectangle? Figure 1.4 is a visual proof of the recursion formula.

Fibonacci’s “discovery” of the sequence that bears his name occurred about fifty years after Hemachandra made his breakthrough. This was probably not a coincidence. Fibonacci, who was educated in North Africa, was quite familiar with Eastern mathematics. His Liber Abaci (1202), in which the Fibonacci sequence appears, also introduced the Indian positional number system to the West. However, his derivation of the Fibonacci sequence from the sizes of successive generations of rabbits is not found in India.

Musical rhythm patterns and the Padovan numbers

The poetic meters Pingala and Hemachandra studied have an analogue in music. Rhythm patterns are sequences of drum hits that overlay a steady pulse, or beat. Notes—groups of beats—play the same role as syllables in poetry. Drummers hit on the first beat of a note
and are silent on the following beats; the length of a note is the number of beats from one hit to the next.

Some types of music, especially dance music, are identified with specific rhythm patterns. Figure 1.5 shows a few examples. Many of these patterns are composed of one- and two-beat notes. Others consist of two- and three-beat notes. The guajira may be familiar as the rhythm of Leonard Bernstein’s “America”, from *West Side Story*.

### Rhythms of one- and two-beat notes
- merengue bell part (Dominican Rep.)
- cumbia bell part (Columbia)
- mambo bell part (Cuba)
- bintin bell pattern (Ghana)
- also bembe shango (Afro-Cuban)

### Rhythms of two- and three-beat notes
- lesnoto (Bulgaria)
- bomba (Puerto Rico)
- guajira (Spain)
- 12-beat clave (Cuba)

Figure 1.5: Dance rhythms

Hemachandra discovered the sequence that counts patterns of one- and two-beat notes. What sequence counts patterns consisting of two- and three-beat notes? Here are the first twelve entries of this sequence, called the Padovan sequence:

| length 
(n) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-------|---|---|---|---|---|---|---|---|---|----|----|----|
| number of patterns 
(Pₙ) | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 4 | 5 | 7 | 9 | 12 |

If Pₙ is the number of n-beat patterns, then Pₙ₊₁ = Pₙ₋₁ + Pₙ₋₂. The proof of this statement is similar to the argument for notes of length one and two. In this case, partition the patterns of length \( n + 1 \) into patterns of length \( n - 1 \) followed by a two-beat note and patterns of length \( n - 2 \) followed by a three-beat note.

Though not nearly as famous as the Hemachandra-Fibonacci sequence, the Padovan sequence has some interesting features. Just as the limit of the ratios of successive Fibonacci numbers is the “golden number” \( \phi = 1.618033988 \ldots \), the limit of the ratios of successive Padovan numbers is the “plastic number” \( p = 1.324717957 \ldots \). (Oddly enough, the Padovan numbers were first discovered by an architect.)
To see why the ratios of successive Padovan numbers converge to the plastic number, observe that
\[
\frac{P_{n+1}}{P_n} = \frac{P_{n-1}}{P_n} + \frac{P_{n-2}}{P_n} = P_{n-2}/P_{n-1}.
\]
Now suppose the limit \(\lim_{n \to \infty} P_{n+1}/P_n\) exists and equals some number \(x\). Take the limit of both sides of the equation as \(n\) goes to infinity:
\[
x = \frac{1}{x} + \frac{1}{x}.
\]
Therefore \(x\) (if it exists) must be a solution to the cubic equation \(x^3 - x - 1 = 0\). The plastic number \(p\) is the only real root of this equation; the other two are complex conjugates. We have now shown that if the sequence \(\{P_{n+1}/P_n\}_{n=2}^\infty\) converges, its limit equals \(p\).

We must now show that the sequence of ratios of Padovan numbers actually has a limit. Let \(q\) and \(r\) represent the complex roots of \(x^3 - x - 1 = 0\). With a little work, you can show that the \(n\)th Padovan number equals
\[
P_n = \frac{p^{n+2}}{3p^2 - 1} + \frac{q^{n+2}}{3q^2 - 1} + \frac{r^{n+2}}{3r^2 - 1}.
\]
(This formula has the flavor of the Binet formula for the Fibonacci numbers, and you can prove it in a similar way.) Since \(|q| = |r| < 1\), the last two terms approach zero as \(n\) goes to infinity, so that \(\lim_{n \to \infty} P_{n+1}/P_n\) equals \(p\).

The Padovan sequence has some other beautiful properties—for example, it is related to a spiral of equilateral triangles in the way the Hemachandra-Fibonacci sequence is related to a spiral of squares (Figure 1.6), and it is closely connected to the Perrin sequence. See [56] for more examples.

**Naming rhythms**

Since there are hundreds of Sanskrit meters, remembering the pattern for any particular meter requires some effort. Although Pingala’s indexing procedure is mathematically impressive, being able to identify the pattern \(\text{SSSISI}\) as “number forty-one in the catalog of six-beat rhythms” is not of much practical use.

Pingala’s best and most well-known solution to this problem involves the following mapping of groups of three syllables to letters:
CHAPTER 1. MATH FOR POETS AND DRUMMERS

To encode the meter \texttt{SSSISI}, begin by breaking it into groups of threes (\texttt{SSSI-SISI}). These groups correspond to the letters \textit{mj}. At this point, you have essentially converted a binary number (base 2) into an octal number (base 8), which does not seem like much progress. However, Pingala had a clever plan. The letters \textit{mj} can be embedded in a word—say, “mojo”—that is more memorable than “number forty-one.” For good measure, you can write a poem in the “mojo” meter than describes the essential characteristics of the meter and includes the word “mojo.” Musicians also use this method for remembering rhythm patterns.

The mathematician Manjul Bhargava observed that this is perhaps the earliest example of an error-correcting code. Typically, an error-correcting code is a sequence that contains encoded information designed to flag errors in transmission (for example, typographical errors). Credit-card numbers and bar codes both have this feature. Each important aspect of a Sanskrit meter is encoded in the poem. In this case, the rhythm of the poem and the name of its meter provide a check on each other.

The history of the Sanskrit system of naming meters does not end with Pingala. Either in his time or later, the nonsense word \textit{yamātārabhānasalagā} came to be used as a way to remember the mapping of triplets of syllables to letters [51]. The word contains long and short syllables (in the English transliteration of Sanskrit, \textit{a} is a short vowel and \textit{ā} is a long vowel):

\[
y-a-mā-tā-rā-jā-bhā-na-sa-la-gām = \texttt{ISSISIII}.
\]

The pattern \texttt{ISSISIII} has the curious property that each string of three syllables occurs
exactly once. For example, the first three syllables form the pattern 1SS, the second through fourth syllables form SSS, and so on. These patterns are named, using Pingala’s table, by their starting syllable (so that ya represents 1SS). However, the number of syllables in a meter does not have to be a multiple of three. The last two syllables, la and gam, are used for the leftovers. It is not known whether Pingala knew this mnemonic for the triplets, or if it was discovered by poets and drummers that came after him.

The mathematician Sherman Stein [55] recognized that the pattern 1SS1S11S is close to being a de Bruijn sequence. A de Bruijn sequence consists of letters drawn from some alphabet such that every “word” of n letters occurs exactly once in the sequence, if we are allowed to “wrap around” from its end to its beginning. The string 1SS1S111 is a de Bruijn sequence for words of three letters in the alphabet {S, 1} (the combinations 111 and 1S involve wrapping around). Figure 1.7 shows one of them on a circle, alongside a de Bruijn cycle for four-letter words.

Although SSS111 is also a de Bruijn sequence, it is not fundamentally different from 1SS1S11, as each sequence produces the three-letter words in the same order, though starting at a different point in the cycle. Using this notion of equivalence, there are only two possible de Bruijn sequences for three-letter words using the alphabet {S, 1}. How can we find the other one? How can we find all the de Bruijn sequences for four-letter words?

Consider the eight three-letter words. A de Bruijn sequence can be thought of as an ordering of these words: the first word is the first three letters in the sequence; the second word is the second through fourth letters, and so on. You probably have noticed that there are some rules about which words can follow each other. For example, 1S1 can be followed by either 1S1 or 1SS. A powerful representation called a directed graph is useful in organizing these possibilities. The vertices of the graph represent states (in this case, three-letter words). Arrows indicate that one state can legally follow another. The graph is shown in Figure 1.8. Any path that visits each vertex exactly once defines a de Bruijn sequence.
The four-letter problem is more complex, since there are twice as many vertices and edges. It is difficult to draw a graph that does not look like spaghetti! There is an clever solution, however: represent each four-letter word by an edge in the graph, as in Figure 1.9. In this case, you now need to find a path that visits each edge exactly once. There are actually sixteen solutions (the general formula for the number of de Bruijn sequences for \( n \)-letter words is \( 2^{2^n-1-n} \)). I’ll leave that to you.
Figure 1.9: A graph representing the de Bruijn sequence problem for combinations of four letters.
Bibliography


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Journal publications:


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Invited Addresses:

“Poverty and polyphony: a connection between music and economics,” Plenary talk, Bridges: Mathematical Connections in Art, Music, and Science, Donostia, Spain, July 2007

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