Distances in perspective.

I started out with the question: How far apart (in real life) are two objects drawn in perspective? Of course, without any extra information, you can’t answer this question, because each point in the picture is the image of a ray in the plan view (a.k.a. “real life”). So I restricted the problem to objects on the floor. In Mathese,

The viewer’s eye is at \((0, 0, -d)\). She stands on the plane \(y = -h\) and draws a picture on the plane \(z = 0\) of what she sees. Let \(\overline{x}\) and \(\overline{y}\) represent the \(x\)- and \(y\)-coordinates in her picture. She sees points \((\overline{x}_1, \overline{y}_1)\) and \((\overline{x}_2, \overline{y}_2)\). If these points are the images of points located on the plane \(y = -h\) in the plan view, what’s the distance between them in the plan view?

Marc called this question the “skid mark” problem, since it’s equivalent to using a photograph of an accident scene to determine the length of skid marks. I looked up more about this on the Web, and it turns out that there’s a whole field called Photogrammetry that’s concerned with measuring distances using photographs.

I structured this report as a series of questions—I haven’t answered all of them, so there’s plenty more to do! In everything that follows, I assume that the viewer is only looking at objects on the plane \(y = -h\).

1. How does the plane \(y = -h\) appear in the viewer’s picture?

   The plane appears as the strip \(-h \leq y < 0\).
2. Find the distance in the plan view between two lines whose vanishing point is the origin in the $xy$-plane.

Since these lines are perpendicular to the picture plane, their distance is equal to the distance between their intersections with the picture plane—that is, the distance between $y = a_1x$ and $y = a_2x$ is $|h/a_1 - h/a_2|$.

3. Find the distance in the plan view between two horizontal lines in the picture plane.

This required more work. I got

$$
\text{distance between } y = y_1 \text{ and } y = y_2 \text{ equals } dh \left| \frac{x_2}{y_2} - \frac{x_1}{y_1} \right|.
$$

4. (Skid marks problem.) Find the distance between $(\pi_1, y_1)$ and $(\pi_2, y_2)$ in the plan view.

Here goes...

$$
\text{distance} = h \sqrt{\left( \frac{x_2}{y_2} - \frac{x_1}{y_1} \right)^2 + \left( \frac{d}{y_2} - \frac{d}{y_1} \right)^2}.
$$

5. Develop geometric constructions to answer questions 2-4.

I like this question better than my earlier ones. The formula in question 4 is nasty and not particularly intuitive (maybe you have an insight?).

(a) To find the distance between two lines perpendicular to the picture plane, measure the distance between their intersections with the line $y = -h$.

(b) If the lines are parallel to the picture plane, do the equivalent of reflecting them about a line at 45° to the picture plane in the plan view—that is, reflect them in any line whose vanishing point is $(±d, 0)$ in the picture. Find the intersections of the two horizontals with the diagonal line, draw rays from the origin through these points, and measure the distance between the rays on the line $y = -h$.

(c) To find the distance between points, use both of these constructions, plus the Pythagorean Theorem.

6. What’s the equation for the image of a circle in the plan view?

This question is left for the reader. (I got tired.)
**Trompe l’œil.** When I was in Italy I was curious about the perspective tricks you sometimes see in Renaissance art. There’s a particularly good example in the ducal palace at Urbino. The walls of a small room are covered with inlaid wood, depicting cabinets full of books, musical instruments, Platonic solids, and other good things. Each wall is designed so that the objects appear to be in perspective. This room, though impressive, isn’t particularly convincing, but I’ve seen some that really work. This sort of thing is called *trompe l’œil*—meaning *trick the eye*. I found some pictures online at http://www.artlex.com/ArtLex/t/trompeloeil.html.

**Student project ideas.** This would be fun to do after watching some clips of movies that use perspective distortions. Each student builds a small box. A hole—the viewpoint—is cut in one side of the box:

![Diagram of a box with a hole](image)

The idea is to paper the walls and floor of the box to create various *trompe l’œil* effects.

1. Paper the sides and floor with checks or some other simple design. For a more complex project, add other elements—a cube, a table, etc. Make the back wall look like an infinite set of repeats of the room you see. In other words, create the “infinite hallway” effect, as in *Charlie and the Chocolate Factory*.

2. *Alice in the Looking Glass*. Same as above, but make the back wall look like a mirror.

3. This looks like a perspective mistake! But maybe the artist really meant the room to look like this. Paper the floor of the room so that this is the image you see.

![Diagram of a checkerboard floor](image)
4. Construct a box so that two identical figures placed inside appear to be different sizes. Hint: you’ll need to build a false bottom and walls to make this work.

I think this last project will be hard...