THE MATHEMATICS OF GAMES OF PURE CHANCE AND GAMES OF PURE STRATEGY

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1. Introduction

On the surface, there's not much difference between the game "Chutes and Ladders" and the game of "Checkers". Both games have simple rules and a clear criterion for victory. But these two children's games represent two separate categories in the mathematical study of games. "Chutes and Ladders" is, of course, a game of pure chance. "Checkers" on the other hand is a game of pure strategy. The mathematical study of games of pure chance is known as the theory of probability while the study of games of pure strategy is called game theory. In these notes, we provide an introduction to these areas of mathematical study.

The roots of probability theory can be traced directly back to the study of parlor games and gambling. Take a pair of dice and roll them 12 times in succession. Should you bet on seeing double sixes at least once? How many rolls are required so that the odds of seeing double sixes (at least) once is better than $50-50$? This type of question arose in gaming circles in France in the 1650s. In a correspondence between perhaps the two most famous thinkers of the day, Blaise Pascal and Pierre Fermat, the question was resolved. In retrospect, this was the first theorem in the modern theory of probability. Today, probability theory is prominent in every imaginable arena: from marketing to politics to medicine. Probabilities have become part of the vernacular as we discuss the stock-market, sports and even the weather.

Game theory, the study of games of strategy, is, in contrast, a much more recent development. While the study of problems of collective action and stratgegy have a long history in philosophy and economics, these questions are not as obviously suitable for mathematical analysis as probability questions. To analyze a game that is not pure chance one must take into account the motivations and strategies of other players — such notions can be difficult to quantify. The first major result in game theory was proved by John von Neumann in the 1920s. Von Neuman, whom many considered the smartest person alive in his day (a day that included Einstein among other luminaries) was also instrumental in the invention of the computer and the atomic bomb. Von Neumann's theorem is called the *MiniMax Theorem* (see Theorem 7.2 below). It ensures the existence of equilibria in zero sum games i.e. games when no cooperation is possible. In the 1950s, John Nash (of A Beautiful Mind fame) proved the existence of equilibria in much more general situations, including many cooperative games. Nash won the 1994 Nobel Prize in Economics for this work. Today, game theory is a vital branch of mathematics, economics and poltical science. The 2005 Nobel Prize in Economics was awarded to two game theorists working in very different areas of the subject.

2. Games with Cards and Dice

We consider a couple of very simple examples of a games of pure chance which can be devised using a pair of fair dice or a deck of fair cards.

To begin, consider the following game: I roll a pair of fair dice. If I roll doubles (i.e., the two dice are equal) you win the game. Otherwise, I win the game. I propose that we make a "friendly" \$5 wager on this simple game. Should you play? Assuming that you are not completely opposed to gambling (or, for that matter, compulsively determined to gamble at any cost!), your decision here should be based on whether or not the game is fair, or better yet, in your favor. In other words, you would like to know the *probability* that I roll doubles. By convention, the probability of an event (such as "I roll doubles") is a number between 0 and 1 whose size measures the likelihood of the event occuring. Here probability 1 means the event is certain to happen, probability 0 means the event is impossible while, say, probability 1/2 means the even will happen, on average, 50% of the time. What is the probability that I roll doubles? To answer this question, note that it is important to treat the dice as separate entities, so we assume one is red and the other blue. We then see that there are 36 possible rolls and 6 of these are doubles as marked with an X in the diagram:

Blue Dice

		1	2	3	4	5	6
	1	X					
	2		X				
Red Dice	3			X			
	4				X		
	5					X	
	6						X

We conclude that the probability of rolling doubles is $6/36 = 1/6$. Since you are only likely to win this bet, on average, one out of six times, you should probably refuse to bet.

We can make this game a little bit more interesting as follows.

Example 2.1. Again, I will roll a pair of dice. If I get doubles, you win while if I get a sum of nine then I win. Otherwise, I will roll again and keep rolling until one of these two outcomes occurs. What is the probability you win at this game? To answer this question, we make a diagram of the 36 possible rolls with X standing for doubles and O for a sum of nine:

Blue Dice

		1	2	3	4	5	6
	1	X					
	2		X				
Red Dice	3			X			O
	4				X	O	
	5				O	X	
	6			O			X

We see that there are 6 ways to roll a doubles and 4 ways to roll a sum of nine. Since all the other possible rolls are irrelevant, we might as well restrict attention to these 10 rolls. We thus see that the probability of doubles is 6/10. This game favors you and you might care to wager on it!

Next, we consider a simple game with a standard deck of 52 cards.

Example 2.2. You draw 2 cards from the top of the deck without replacement. You win \$5 if you draw a pair of Jacks. Otherwise you pay me \$5. What are your chances in this game?

This game brings up an important issue. How should we think of a 2-card hand? On the one hand, we may wish to keep track of the order we received the cards. Thus a 2-card hand is an ordered pair (1st Card, 2nd Card) of 2 cards. With this view, we will distinguish, for example the outcome

$$\text{outcome } 1 = (\text{1st Card} = A\diamondsuit, \text{2nd Card} = J\spadesuit)$$

from

$$\text{outcome } 2 = (\text{1st Card} = J\spadesuit, \text{2nd Card} = A\diamondsuit).$$

How many possible 2-card hands are there? We have 52 possibilities for the 1st Card, and having chosen this card, 51 possibilities remaining for the 2nd Card giving $52 \cdot 51 = 2652$ possible ordered 2-card hands.

How many of these 2652 outcomes will win us \$5? To get a pair of Jacks, first we must choose our 1st Card to be a Jack: there are 4 choices. Then we must choose our 2nd Card to be another Jack: there now are 3 choices left. We conclude that $4 \cdot 3 = 12$ of the 2652 possible ordered 2-card hands will make us a winner. Our chances of winning the game are not good: Since $12/2652 = 1/221$, we can expect to win this game only once out of every 221 times we play.

There is an important alternative way to think of 2-card hands and to do the preceding calculation. In many card games (including this one) it is not the order in which you receive your cards that matters but simply the cards themselves. It is natural then to view the possible outcomes of this game as *sets* { 1st Card, 2nd Card } of 2 cards. We can then rely on the customary meaning of a set of objects and, particularly, what it means to say two sets are equal. Recall that two sets are equal if they have the same elements, regardless of the order or manner in which the elements are described. For example, the set of integers larger than 2 and the set of positive integers whose square is bigger than 4 are the same set. In our case, we can write

$$\text{outcome} = \{J\spadesuit, A\diamondsuit\}$$

for both of the ordered 2-card hands above.

How many 2-card hands are there? We could count the hands directly. Or we can simply observe that there are 1/2 as many outcomes as in the ordered case since each unordered set of 2 cards gives rise to exaclty 2 ordered hands. Thus there are $1/2 \cdot 52 \cdot 51 = 1326$ different outcomes.

How many of the 2-card hands consist of two Jacks? We can list them:

$$\{J\spadesuit, J\heartsuit\}, \{J\spadesuit, J\diamondsuit\}, \{J\spadesuit, J\clubsuit\}, \{J\heartsuit, J\diamondsuit\}, \{J\heartsuit, J\clubsuit\}, \{J\diamondsuit, J\clubsuit\}.$$

We could also have simply observed that there are 1/2 as many 2-element subsets as ordered pairs. In any case, in this setting we have 6 ways to win out of 1326 equally likely outcomes putting our chances, again, at 1/221.

These simple examples illustrate several features of probability theory that we will explore in these notes. Example 2.2 shows that computing probabilities is related to counting. In Section 4, we will explore this aspect of the theory in more detail. Example 2.1 illustrates a first calculation of what is known as *conditional probability*. We define this concept in Section 3 and use it to analyze the dice game "Craps" in Section 5.

Finally, we remark that our probability calculations gives rise to a precise measure of how much a game can be expected to pay off (or cost) if played repeatedly over time. This quantity is called *expected value* of the game and is defined as follows:

Definition 2.3. Suppose a game of pure chance has n possible outcomes with probabilities $p_1, p_2, \dots p_n$ (so that $p_1+p_2+\cdots+p_n = 1$) and payoffs $C_1, C_2, \dots, C_n$. Then the *expected vallue* of the game is defined to be the value

$$\mathrm{E} = C_1 \cdot p_1 + C_2 \cdot p_2 + \cdots + C_n \cdot p_n.$$

For example, suppose we play the dice game in Example 2.1 100 times. How much money should you expect to win? Well, you win with probability $p = .6$ and lose with probability $1 - p = .4$. Thus the expected value is

$$\mathrm{E} = 5(.6) - 5(.4) = 1.$$

The expected value calculation tells you that you can expect to win \$1 on each repetition. Note that you can't actually win \$1 on any repetition, this is just an average! Thus if we play 100 times you can expect to win \$100. Compare this with the card game (Example 2.2) in which $\mathrm{E} = 5(1/221 - 220/221) = -4.95$. You will lose your shirt if you play this game repeatedly. Expected values provide a (theoretical) cash register for the outcome of wagering. We conclude with one further example.

Example 2.4. Consider the following game. You roll the dice and I pay you the dollar amount of the roll. Sounds like fun, right? But what if I charge you \$5 to play? Is it still a reasonable game to play? To answer these question, we simply compute the expected value of the game. Note that the outcomes are now "Sum $= n$" for $n = 2, 3, \dots, 12$. We write $P(n)$ for the probability of "Sum $= n$". Using the dice table above, we calculate these probabilities and compute

$$\begin{aligned}
\mathrm{E} &= 2 \cdot P(2) + 3 \cdot P(3) + 4 \cdot P(4) + 5 \cdot P(5) + 6 \cdot P(6) + 7 \cdot P(7) \\
&\quad + 8 \cdot P(8) + 9 \cdot P(9) + 10 \cdot P(10) + 11 \cdot P(11) + 12 \cdot P(12) \\
&= 2 \cdot (1/36) + 3 \cdot (2/36) + 4 \cdot (3/36) + 5 \cdot (4/36) + 6 \cdot (5/36) + 7 \cdot (6/36) \\
&\quad + 8 \cdot (5/36) + 9 \cdot (4/36) + 10 \cdot (3/36) + 11 \cdot (2/36) + 12 \cdot (1/36) \\
&= 252/36 \\
&= 7.
\end{aligned}$$

Thus we expect to make \$7 per roll on the first dice game and \$7 - \$5 = \$2 per roll on the second. Alternately, this calculation shows that 7 is the average roll of a pair of dice, an unsurprising fact.

Exercises for Section 2

Exercise 2.1. Compute the expected value of the following game: You roll a pair of dice. I pay you the dollar amount of your roll if it is doubles or a sum of seven, otherwise you pay me the dollar amount of the sum.

Exercise 2.2. Consider the following game which costs \$8 to play. Throw three dice. I pay you \$5 for each dice that shows a 6. What is the expected value of this game?

Exercise 2.3. Compute the expected value of the following game. You flip a coin. If you get heads, I give you \$2. The game is over. If not, you flip again. If you get heads on the 2nd flip, you get \$4 and, again, the game is over. If not flip a third time. If you get heads, you win \$8 and we're through. If you get tails, keep flipping. In general, I will pay you $\$2^n$ if you get heads for the first time on the nth flip for $n = 4, 5, 6, 7, 8$. However, if you get 9 tails in a row, you get nothing.

Exercise 2.4. Consider the following variation on the previous game. You flip a coin. If you get heads, I give you \$2. The game is over. If not, you flip again. If you get heads on the 2nd flip, you get \$4 and, again, the game is over. If not flip a third time. If you get heads, you win \$8 and we're through or else you keep flipping. In general, I will pay you $\$2^n$ if get heads for the first time on the nth flip. However, this time you do not stop flipping after 9 tries but keep flipping the coin until you get heads. How much money would you be willing to pay up front to play this game? Please justify your answer with a mathematical argument.

3. Basic Notions and Laws of Probability

In this section, we state precise definitions of the basic notions of probability theory and then give some first results concerning these definitions.

By a *probability experiment* we will mean any action or process whose execution results in exactly one of a number of well-determined and equally likely possible outcomes. We would like to think of rolling a dice, flipping a 2coin and dealing a hand of cards as examples of probability experiments. Note that we must be careful about describing the outcomes so that they are all equally likely. For example, when rolling a pair of dice we should agree that there are 36 possible outcomes as in Figure 1 below.

The *sample space* S of a probability experiment is the set of all possible outcomes. An *event* E in a probability experiment is a particular subset of the sample space. That is E is a particular, distinguished set of outcomes. We will often say an outcome in E is *favorable* to E. Finally, we define the *probability* $P(E)$ of E to be the fraction of the total number of outcomes in S that are in E. Precisely, any set A let

$$n(A) = \text{ the number of elements in } A.$$

Then we define

$$P(E) = \frac{n(E)}{n(S)}.$$

Example 3.1. Consider, as in the introduction, the probability experiment of rolling a pair of dice. The sample space S for this experiment is then the 36 possible rolls indicated in Figure 1. Let E be the event $E=$ "the roll is doubles". Then $n(E) = 6$ and so $P(E) = 6/36)$. Similarly, if $F =$ "the sum is nine" then $n(F) = 4$ and $P(F) = 4/36)$.

Example 3.2. Consider the experiment of drawing 2 cards from a standard deck of 52 cards. In this case, we have two possible ways to view the sample space as explained in the introduction. We may choose to think of the sample space as ordered pairs of 2 cards. In this case, the sample space contains $52 \cdot 51 = 2652$ different outcomes. Alternately, we may decide to treat each set of 2 cards as an outcome. In this case, our sample space is the collection of all 2-element subsets of the 52 cards. A formula for the number of (in general) k-element subsets from an n-element set will turn out to be a critical tool in our work in Section 4. In this case, we have observed that the sample space has $1/2 \cdot 52 \cdot 51 = 1326$ different outcomes.

Suppose now that we have a probability experiment with sample space S. Given two events E_1 and E_2 we can construct new events using the conjunctions "and" and "or" and the negation "not". Specificaly,

$$E_1 \text{ or } E_2 = \text{all outcomes favorable to either } E_1,\ E_2 \text{ or both}$$

$$E_1 \text{ and } E_2 = \text{all outcomes favorable to both } E_1 \text{ and } E_2 \text{ simultaneously}$$

$$\text{not } E_1 = \text{all outcomes unfavorable to } E_1$$

For example, in our experiment of rolling a pair of dice, we might let $E_1 =$ "roll is doubles" and $E_2 =$ "sum is 8". To compute $P(E_1 \text{ or } E_2)$, we should count all the doubles (6) and all the eights (5) and add $6 + 5 = 11$. However, double 4's was counted twice and so we should subtract 1. Thus $n(E_1 \text{ or } E_2) = 11 - 1$ and $P(E_1 \text{ or } E_2) = 10/36)$. Notice here that E_1 and $E_2 =$ "roll is double 4's" is precisely the event we subtracted off. it is even easier, to compute $P(\text{not } E_1)$. Since there are 6 doubles there must be $36 - 6 = 30$ non-doubles. Thus $P(\text{not } E_1) = 30/36) = 1 - P(E_1)$. This examples generalizes to

Theorem 3.3. Let E_1 and E_2 be events in a probability experiment. Then

(1) $P(E_1 \text{ or } E_2) = P(E_1) + P(E_1) - P(E_1 \text{ and } E_2)$

(2) $P(\text{not } E_1) = 1 - P(E_1)$

Proof. The proofs are counting arguments which can be visualized with Venn diagrams. Let S be the sample space. For (1) we have

$$\begin{aligned} P(E_1 \text{ or } E_2) &= \tfrac{n(E_1 \text{ or } E_2)}{n(S)} = \tfrac{n(E_1)+n(E_2)-n(E_1 \text{ and } E_2)}{n(S)} \\ &= P(E_1) + P(E_2) - P(E_1 \text{ and } E_2). \end{aligned}$$

For (2) we have

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{n(S) - n(\text{not } E_1)}{n(S)} = 1 - P(E_1)$$

□

We say two events E_1 and E_2 are *mutually exclusive* if $P(E_1 \text{ and } E_2) = 0$. Thus mutually exclusive events are those that cannot happen simultaneously, like $E_1 =$ "roll is doubles" and $F =$ "roll is nine" from Figure 1. Theorem 3.3 (1) yields the following

Corollary 3.4. *(The Law of Mutually Exclusive Events)* Let E_1 and E_2 be mutually exclusive events. Then

$$P(E_1 \text{ or } E_2) = P(E_1) + P(E_2)$$

Corollary 3.4 answers the question: "When do we add probabilities?" Answering the related question "When do we multiply probabilities?" leads to a critically important concept in probability theory: namely, *conditional probability.*

To introduce the notion of conditional probability, consider the following simple game. I roll a pair of dice so that you cannot see. I tell you what the sum of the dice is, e.g. "sum = 6". You win $5 if the roll is doubles otherwise you lose. The question we ask is: "What effect does the information I give you have on your willingness to play?" Let's examine this question in some special cases. Suppose I tell you the "sum is 2". Then you are guaranteed to have doubles! We will say the probability "roll is doubles" *given* the "sum is two" is 1. Notationally, this is written

$$P(\text{ "roll is doubles" } \mid \text{ sum is 2 }) = 1.$$

Here the symbol "|" reads as the word "given".

Let's compute in some other special cases: Suppose I tell you the "sum is 6". Then we may as well restrict our attention to the 5 ways to make a six. That is, writing B and R for the Blue and Red dice, our sample space can be effectively reduced to

$$\{B = 1, R = 5\}, \{B = 2, R = 4\}, \{B = 3, R = 3\}, \{B = 4, R = 2\}, \{B = 5, R = 1\}.$$

Now of these six rolls, only one is favorable to "roll is doubles". Thus

$$P(\text{ "roll is doubles" } \mid \text{ sum is 6 }) = 1/5.$$

The chances of winning are not so great but notice that they are slightly higher than they would be without the information "sum is 6". Contrast this with the case "sum is 7". There are now 6 rolls with sum 7 but obviously none are doubles. Thus

$$P(\text{ "roll is doubles" } \mid \text{ sum is 7 }) = 0.$$

Such examples suggest how to compute $P(E_1|E_2)$. First we restrict our attention the outcomes favorable to E_2. In other words, E_2 becomes the sample space since we know that one of these outcomes certainly occurred. We then count the events in E_2 favorable to E_1 also – that is we find $n(E_1 \text{ and } E_2)$. We then divide:

$$P(E_1|E_2) = \frac{n(E_1 \text{ and } E_2)}{n(E_2)}. \tag{1}$$

From this formula we can prove:

Theorem 3.5. *(Law of Conditional Probability)* Let E_1 and E_2 be events in a probability experiment. Then

$$P(E_1 \text{ and } E_2) = P(E_1|E_2) \cdot P(E_2).$$

Proof. To prove this result, we "unsimplify" the fraction in expression (1) to create a complex fraction:

$$\begin{aligned} P(E_1|E_2) &= \frac{n(E_1 \text{ and } E_2)}{n(E_2)} \\ &= \frac{n(E_1 \text{ and } E_2)}{n(E_2)} \cdot \frac{\frac{1}{n(s)}}{\frac{1}{n(s)}} = \frac{\frac{n(E_1 \text{ and } E_2)}{n(S)}}{\frac{n(E_2)}{n(S)}} = \frac{P(E_1 \text{ and } E_2)}{P(E_2)} \end{aligned}$$

We then clear the denominator to get the desired formula. □

We can use Theorem 3.5 to give a satisfactory answer to the question "When do we multiply probabilities?" To do so we must introduce the notion of independent events. Informally, two events are independent events if the occurance of one has no effect on the occurance of the other. For example, if I flip a coin and then roll a pair of dice the events "coin is heads" and "roll is doubles" are independent.

For another, consider the experiment of drawing a card from a deck of 52. Let E_1 = "Jack" and E_2 = "Red card". then $P(E_1) = 1/13$. If E_2 occurs i.e. the card is red, then the chance of a Jack is still $2/26 = 1/13$. That is, $P(E_1|E_2) = P(E_1)$. We make this our formal definition:

Let E_1 and E_2 be two events in some probability experiment. We say E_1 and E_2 are *independent events* if $P(E_1|E_2) = P(E_1)$.

We can now answer our question "When do we multiply probabilites?" with the following theorem:

Theorem 3.6. *(The Law of Independent Events)* Let E_1 and E_2 be independent events. Then

$$P(E_1 \text{ and } E_2) = P(E_1) \cdot P(E_2)$$

Proof. We just use Theorem 3.5 to write

$$P(E_1 \text{ and } E_2) = P(E_1|E_2) \cdot P(E_2)$$

and then substitute $P(E_1) = P(E_1|E_2)$ since E_1 and E_2 are independent events. □

We have now developed the basic laws of probability. We give some examples of how these simple ideas can be put to work.

Example 3.7. I have 10 fair dice. I will roll them simultaneously and give you \$5 if at least one of the dice comes up a 6. Otherwise, you will owe me \$5. Should you play? We will need to compute probabilities to decide. Note that the outcome of the roll of each of the 10 dice is independent of the other 9 dice. More precisely, let E_1= "dice 1 is not a 6", E_2 = "dice 2 is not 6", ..., E_{10} = "dice 10 is not 6". Then

$$P(E_1) = P(E_2) = \cdots = P(E_{10}) = 5/6.$$

Moreover, $E_1, E_2, \ldots, E_{10}$ are mutually independent events. Thus

$$\begin{aligned} P(\text{“no dice is a 6”}) &= P(E_1 \text{ and } E_2 \text{ and } \cdots \text{ and } E_{10}) \\ &= P(E_1) \cdot P(E_2) \cdots P(E_{10}) \\ &= (5/6)^{10} \\ &= .1615 \end{aligned}$$

Thus

$$P(\text{“at least one six”}) = 1 - P(\text{“no dice is a 6”}) = 1 - .1615 = .8385.$$

The expected value of the game is E $= 5 \cdot P(\text{“at least one six”}) - 5(1 - P(\text{“at least one six”}) =$ 3.385. You could expect to make about \$3.38 per roll.

Example 3.8. What is the probability of rolling 6 dice simultaneously and getting each of the possible numbers 1 through 6 as your roll? We might call such a roll a "straight". To answer this question, it is useful to imagine the dice are all different collors, say Red, Blue, Green, Orange, Purple and White. Now one way of obtaining a straigh is to have $R = 1, B = 2, G = 3, O = 4, P = 5$ and $W = 6$. Let's call this straight "RBGOPW". What is the probability of getting this particular straight? We have

$$\begin{aligned} P(\text{“RBGOPW”}) &= P(R = 1 \text{ and } B = 2 \text{ and } \cdots \text{ and } W = 6) \\ &= P(R = 1) \cdot P(B = 2) \cdots P(W = 6) \\ &= (1/6)^6 \\ &= .00002 \end{aligned}$$

But "RBGOPW" is only one way to get a straight. We could also have "RBGOWP" and "BRGOPW" and many others. However, each of these straights have the same probability, namely .00002. Moreover, these events are all mutually exclusive. Thus $P(\text{“straight”}) = N \cdot (.00002)$ where N is the number of "straights". In Section 4, we find a formula for the number of ways to order k letters or objects – the number of *permutations* of a set. The answer may be familiar to you. Ask: How many ways to order the 6 letters R, B, G, O, P, W? Answer: First choose the first letter (6 choices), then the second letter (5 choices), then the third letter (4 choices) etc. until we have all 6 in order. Thus here are $N = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ orderings. This is often denoted $N = 6!$ (6 factorial). We conclude $P(\text{“straight”}) = 6! \cdot (.00002) = .0144$.

Exercises for Section 3

Exercise 3.1. Compute the following probabilities for the experiment of rolling a pair of dice:

a. $P($ "doubles" and "sum at least 8"$) =$

b. $P($ "sum is odd" or "at least one dice is a 5"$) =$

c. $P($ "sum is at least 8" | "at least one dice is a 5"$) =$

d. $P($ "sum is 6" | "doubles"$) =$

Exercise 3.2. Compute the following probabilites for the experiment of drawing two cards (without replacement) from a deck of 52:

a. $P($ " a pair"$) =$

b. $P($ "both cards red"$) =$

c. $P($ "a pair" | "both cards red"$) =$

d. $P($ "both cards red" | "a pair" $) =$

Exercise 3.3. Determine the probability of throwing a "straight" with 7 dice i.e. of throwing 7 dice and having the numbers 1 through 6 (plus one more number) appear.

Exercise 3.4. Use the ideas developed so far to solve "Pascal's problem" mentioned above: Roll the dice once. What is the probability you *don't* get double sixes? Roll the dice twice in a row. What is the probability you don't get double sixes on either roll? Now generalize: Determine the number of rolls so that the probability of seeing double sixes on at least one roll is greater than .5.

4. Counting Cards in Bridge and Poker

Our ability to compute the probabilities which arise in simple games is often directly related to our ability to *count* occurances of various outcomes in rather complicated situations. The casino scene in the film Rain Man illustrates the possibilities for "counting cards" and winning at, in this case, Blackjack. A fast and precise photographic memory, as the character Raymond has in the movie, can improve the odds of winning over-all at Blackjack under casino rules to better than even. To win money, it is not sufficient just to be able to count the cards. Knowledge of the probabilities which arise as a shoe of cards is played out is critical to know when and how to bet. Here there is also "counting" going on but it is of a mathematical nature.

Consider the Poker game of 5-Card Draw. We will simplify matters by assuming that you are playing alone and that you are dealt 5 cards from a standard deck of 52. (A boring game yes, but not much worse than playing a slot machine!) A winning hand for you is a pair of Jacks or better but you need not know what this means yet. We are faced with a counting question regarding the possible 5-card hands. Again the issue of "to order" or not "to order" arises as in Example 3.2 above.

Let's begin by ordering, i.e., by keeping track of the order in which we receive our five cards C_1, C_2, C_3, C_4 and C_5. A particular hand then might be

$$C_1 = A\spadesuit, C_2 = 10\diamondsuit, C_3 = 2\diamondsuit, C_4 = Q\clubsuit, C_5 = Q\heartsuit.$$

This is a different hand from, say,

$$C_1 = Q\heartsuit, C_2 = 10\diamondsuit, C_3 = 2\diamondsuit, C_4 = Q\clubsuit, C_5 = A\spadesuit.$$

How many ordered 5-card hands are there? We can build one by first choosing C_1 (52 choices), then C_2 (51 choices), then C_3 (50 choices), then C_4 (49 choices) and then C_5 (48 choices). Every such sequence of choices yields a distinct hand. Thus we see that there are $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$ total ordered 5-card hands. Let us write $P(52, 5) = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$ and, more generally,

$$P(n, k) = n \cdot (n-1) \cdots (n-k+2) \cdot (n-k+1)$$

for $n \geq k \geq 1$. Then $P(n, k)$ counts the number of ways to build an ordered list of k objects from a set of n objects. The special case $n = k$ recovers the factorial function: $P(n, n) = n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$. This is simply the number of ways to order a set of n objects or the number of *permutations* of n objects.

Suppose we choose instead to view a 5-card hand without regard to order. We can then write both hands above (and many more like them) as the *set*

$$\{A\spadesuit, 10\diamondsuit, 2\diamondsuit, Q\clubsuit, Q\heartsuit\}.$$

We wish to count the number of 5-element subsets which can be chosen from the set of 52 cards. The formula in this case is not much harder but the answer is quite important. Observe that $P(52, 5)$ over-counts the number of unordered 5-card hands since, for example, the two ordered hands above are now regarded as the same. But here's a key question: How many ordered hands does $\{A\spadesuit, 10\diamondsuit, 2\diamondsuit, Q\clubsuit, Q\heartsuit\}$ represent? Certainly any ordering of these 5-cards will give a distinct ordered hand. Moreover, any ordered hand with these cards comes from one such ordering. Thus there are exactly $P(5, 5) = 5!$ ordered hands for every single unordered hand. We conclude that

$$\text{number of unordered 5-card hands} \quad = \frac{P(52, 5)}{P(5, 5)} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2,598,960.$$

In general, we define

$$\binom{n}{k} = \frac{P(n, k)}{P(k, k)} = \frac{n \cdot (n-1) \cdots (n-k+2) \cdot (n-k+1)}{k \cdot (k-1) \cdots 2 \cdot 1}.$$

The numbers $\binom{n}{k}$ are usually called the *binomial coefficients* because of their role as the coefficients in the expansion of binomials of the form $(x + a)^n$. They may also be familiar from Pascal's triangle. In fact, it was the gambling question discussed in the introduction (Exercise 2.4) which led to their discovery by Pascal. We will refer to the number $\binom{n}{k}$ as n *choose* k to highlight the role of these numbers in counting. We have seen that $\binom{n}{k}$ counts the number of k-element subsets that can be chosen from a set of n elements.

Example 4.1. Ten students meet in the fieldhouse to have a game of pick-up basketball. How many ways can two teams be chosen? Since there are 10 players and 5 per team, each time we choose one team the other is determined as well. So how many ways are there to choose a 5 player team from 10 players? The answer is

$$\binom{10}{5} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 252 \quad \text{ways.}$$

The game of Bridge is played with four players and a standard deck of 52 cards. The cards are dealt out completely so that each player has 13. We won't describe the play of the game, which would take us very far afield. We simply mention that the number of cards in each suit that you have in your hand becomes very important in Bridge. For example, if all 13 of your cards are the same suit (a very improbable event!), you can make a "Grand Slam". Suppose we draw 13 cards from a fair deck and find we have $S = 5$ spades, $H = 4$ hearts, $D = 2$ diamonds and $C = 2$ clubs. Note that the suit counts necessarily add up to 13. These numbers describe the *distribution* of our hand.

Example 4.2. We can ask a variety of probability questions concerning distributions of bridge cards. Let's consider the following one here: What is the probability of having exactly 8 cards in one suit in a 13 card bridge hand? Our formula for probabilities gives:

$$P(\text{"8 cards in some suit"}) = \frac{\text{number of hands with 8 cards in a suit}}{\text{number of 13 cards hands}}.$$

We know the denominator: There are $\binom{52}{13}$ 13-card hands. Notice that we are using *unordered* hands here. This will usually be our preference. We can compute the numerator, as follows: First we'll count the hands with exactly 8 spades, i.e. with S = 8. How do we build such a hand? First we pick the 8 spades from the 13 available – there are $\binom{13}{8}$ ways to pick do this. Next we need 5 more cards and none of them should be spades. So let's throw all the spades out of the deck leaving $52 - 13 = 39$ cards. We'll then choose our remaining 5 cards from these 39 – there are $\binom{39}{5}$ ways to do this. Thus there are $\binom{13}{8} \cdot \binom{39}{5}$ hands with exactly 8 spades. Finally, we observe that there are the same number of hands with exactly 8 hearts, with 8 diamonds and with 8 clubs. Thus we should multiply by 4 to get the number of hands with 8 in one suit. In conclusion,

$$P(\text{"8 cards in some suit"}) = \frac{4 \cdot \binom{13}{8} \cdot \binom{39}{5}}{\binom{52}{13}} = .00467.$$

Let's return to the game of Poker. As you may know, there are names for certain special hands: In ascending order of strength, we have *One Pair*, *Two Pair*, *Three of a Kind*, *Straight* (5 cards in ascending order, Aces can be low or high) *Flush* (all 5 cards the same suit), *Full House* (3 of a kind plus a pair) *Four of a Kind* and *Straight Flush* (5 cards in succession, all of the same suit). Of course, some 5-card hands could go by different names. For example, a Four of a Kind hand could be thought of as Two Pair. We will remove this ambiguity by always using the strongest name for a hand possible where strength is measured by the order above.

This brings us to an interesting question: Why are the 5-card Poker Hands ordered in this way? You may suspect the answer has to do with probabilities. As we will see, our list of Poker hands is in decreasing order of probability: That is, we have:

Theorem 4.3.

$$P(\textit{One Pair}) > P(\textit{Two Pair}) > \cdots > P(\textit{Four of a Kind}) > P(\textit{Straight Flush})$$

Proof. The proof is obtained by computing each of the probabilities in question directly. We will take this up in the next example and in the exercises. □

Example 4.4. We compute the probabilities of some of the special hands and leave the remaining for the exercises.

Straight Flush: This one is the easiest of all. A Straight Flush is determined by a suit (4 choices) and then the lowest card which can be an Ace through a 10, since 10 J Q K A is the highest straight, (10 choices). Thus there are 40 total straight flushes and

$$P(\text{Straight Flush}) = \frac{40}{\binom{52}{5}} = .000015.$$

Flush: How do we describe a Flush? We first specify what suit all of our cards are (4 choices) and then say what 5-cards of this suit we have. Now once we fix a suit, we are restricted to 13 card values. Thus there are $\binom{13}{5}$ possible flushes in each fixed suit and $4 \cdot \binom{13}{5}$ total flushes. Now we must be careful, since we don't want to count the straight flushes again. However, we know there are 36 of these and so we can subtract to get $4 \cdot \binom{13}{5} - 40$ flushes. We conclude:

$$P(\text{ Flush}) = \frac{4 \cdot \binom{13}{5} - 40}{\binom{52}{5}} = .00197.$$

Three of a Kind: As in the case of a flush, it is useful to imagine describing a typical hand (in this case, a Three of a Kind) to try to count all such hands. So how would we describe a hand with Three of a Kind? First we might say which card value (2 through Ace) we have three of: there are 13 possibilities here. Next we might say which suits our three are in: there are 4 suits and we need to choose 3 so there are $\binom{4}{3} = 4$ possibilities here. Finally, we will indicate our last two cards. Here we must be a little careful. We don't want either to be the 4th remaining card of our three of a kind. Thus we should reduce our choices to the 48 cards which remain, when we exclude these four. Now we also don't want to get a pair when we choose 2 cards from these 48, for then we would have a Full House. So we ask how many ways are there to choose two cards but not a pair from 48? Well there are $\binom{48}{2}$ possible 2-card hands. How many are pairs? To get a pair, we first pick one of the remaining 12 values and then choose the two suits (4 suits, choose 2 or $\binom{4}{2} = 6$ choices). Thus there are $\binom{48}{2} - 12 \cdot 6 = 1056$ ways to choose the last two cards. We conclude that

$$P(\text{Three of a Kind}) = \frac{4 \cdot 13 \cdot 1056}{\binom{52}{5}} = .02113.$$

Exercises for Section 4

Exercise 4.1. Compute the probability of drawing 5 cards and getting Four of a Kind.

Exercise 4.2. Compute the probability of drawing 5 cards and getting a Full House.

Exercise 4.3. Compute the probability of drawing 5 cards and getting a Straight

Exercise 4.4. Compute the probability of drawing 5 cards and getting Two Pair.

Exercise 4.5. Compute the probability of drawing 5 cards and getting One Pair.

Exercise 4.6. Compute the probability of having two 6 card suits in a hand of Bridge.

Exercise 4.7. Compute the probability of having one 9 card suit in a hand of Bridge.

5. Probability Trees and the Game of Craps

Many games have branching features wherein the outcome of one stage of the game affects the odds in the subsequent stages. A famous example of this is the game Craps in which the opening roll or point determines the criteria for winning. (We will examine this game carefully below.) The idea of a branching game can be applied to tournaments as well. For example, in the NCAA basketball field of 64 we may think of each round as a stage in a large game. A team's chances of winning at each stage is determined not just by their strength but by the opponents they meet at each stage.

To understand the probabilities in a game with this type of branching it is necessary to consider conditional probabilities as discussed in Section 4. Fortunately, we have an excellent device at our disposal for keeping things organized – a *probability tree.*

We illustrate the use of a probability tree, by considering a simple game. You flip a coin twice. Let H be the number of heads you get so that $H = 0, 1$ or 2. I give you one dice if $H = 0$, two dice if $H = 1$ and three dice if $H = 2$ and you roll your dice. You win \$5 if the sum of the dice you throw equals 5 and lose \$5 with any other sum.

To compute the probability of winning this game, it helps to consider the various possibilities for the coin toss individually. Suppose we throw no heads so that $H = 0$. Then we only have one dice to roll and our chances of getting a 4 are 1/6. In the language of conditional probability we can say

$$P(\text{“sum} = 5\text{”} | \text{“}H = 0\text{”}) = 1/6.$$

If $H = 1$ then we throw two dice and our chances of getting a sum of 5 is $4/36) = 1/12$. Thus

$$P(\text{“sum} = 5\text{”} | \text{“}H = 1\text{”}) = 1/12.$$

What if $H = 2$? Then we roll three dice. There are now 3 ways to get a sum of 5 $= 2 + 2 + 1$ (i.e. in which 2 of the dice are a 2 and the other a 1) and also 3 ways to get a sum of $5 = 3 + 1 + 1$. Since there are 6^3 possible rolls for three dice we conclude

$$P(\text{“sum} = 5\text{”} | \text{“}H = 2\text{”}) = 6/6^3 = 1/36).$$

So we know the probability of winning the game in each of the possible scenarios, but how do convert this into a probability of winning outright? Notice that, logically, a winning outcome must belong to one of the three events: "sum = 5" and "$H = 0$", "sum = 5" and "$H = 1$, or "sum = 5" and "$H = 2$. Morever, these events are mutually exclusive. Thus

$$\begin{aligned} P(\text{"sum is 5"}) = P(&\text{"sum = 5" and "}H = 0\text{"}) \\ &+ P(\text{"sum = 5" and "}H = 1\text{"}) \\ &+ P(\text{"sum = 5" and "}H = 2\text{"}). \end{aligned}$$

Now the event "sum is 5" and the event "$H = 0''$ (or $H = 1, 2$) are certainly not independent events. By design, the probability of "sum is 5" changes depending on the value of H. However, we have the formula:

$$P(E_1 \text{ and } E_2) = P(E_1|E_2) \cdot P(E_2)$$

(Theorem 3.5). Here this becomes

$$\begin{aligned} P(\text{"sum = 5" and "}H = 0\text{"}) &= P(\text{"sum = 5"}|\text{"}H = 0\text{"}) \cdot P(\text{"}H = 0\text{"}) \\ P(\text{"sum = 5" and "}H = 1\text{"}) &= P(\text{"sum = 5"}|\text{"}H = 1\text{"}) \cdot P(\text{"}H = 1\text{"}) \\ P(\text{"sum = 5" and "}H = 2\text{"}) &= P(\text{"sum = 5"}|\text{"}H = 2\text{"}) \cdot P(\text{"}H = 2\text{"}). \end{aligned}$$

Thus

$$\begin{aligned} P(\text{"sum is 5"}) = P(&\text{"sum = 5"}|\text{"}H = 0\text{"}) \cdot P(\text{"}H = 0\text{"}) \\ &+ P(\text{"sum = 5"}|\text{"}H = 1\text{"}) \cdot P(\text{"}H = 1\text{"}) \\ &+ P(\text{"sum = 5"}|\text{"}H = 2\text{"}) \cdot P(\text{"}H = 2\text{"}). \end{aligned}$$

It is straightforward to compute $P(\text{"}H = 0\text{"}) = 1/4, P(\text{"}H = 1\text{"}) = 1/2$ and $P(\text{"}H = 2\text{"}) = 1/4$ and so we conclude:

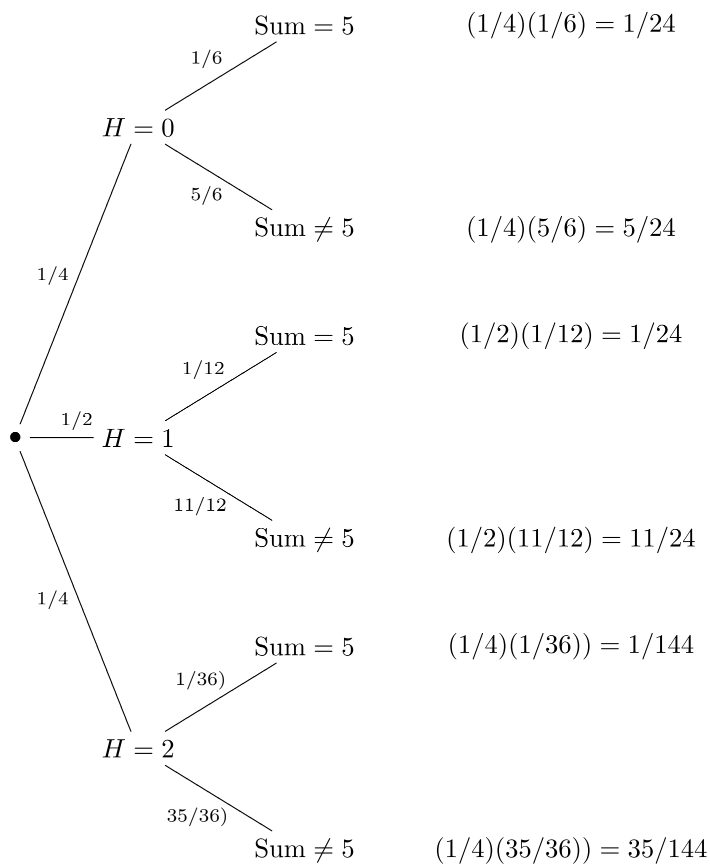
$$\begin{aligned} P(\text{"sum is 5"}) &= (1/4) \cdot (1/6) + (1/2) \cdot (1/12) + (1/4) \cdot (1/36)) \\ &= .0903 \end{aligned}$$

We can visualize this calculation with a probability tree (Figure 2, below). The tree opens from left to right with the root node $\bullet$ representing the beginning of the game. The first event is the flip of the coin. We follow *branches* to the three possible outcomes or *nodes* $H = 0, 1$ or 2 which can occur here. We label the branches with the probabilities of reaching the nodes.

The next stage is the roll of the dice. In this case, we are only interested in two possible outcomes of this roll: Sum = 5 or Sum $\neq$ 5. We label the branches to these two types of nodes with the probabilities of reaching them. Notice it is here that we are implicitly looking at conditional probabilities. For example, the probability of reaching the node Sum = 5 from the node $H = 0$ is precisely the conditional probability $P(\text{"sum = 5"}|\text{"}H = 0\text{"}) = 1/6$.

The probability of traversing any path from left to right in the tree is obtained by multiplying the branches. Thus, for example, the probability that the game will result in you throwing 2 Heads and then rolling a sum other than five with three dice is 35/144. The tree facilitates correct and easy use of the Law of Conditional Probability. It is easy to imagine using a tree for branching games with three or more stages Notice, however, that the number of nodes of the tree grows exponentially with the number of stages and so writing down a complete tree quickly becomes unfeasible.

Figure 2: A Probability Tree



We can put the probability tree to good use in analyzing a the game "Craps". Played on a long green velvet table with one roller and many side betters and spectators, "Craps" is the classic casino dice game. The game can go on indefinitely, in principle, with the tension and side bets mounting with each throw. The rules are simple: The roller throws the dice. If she rolls a sum of 7 or 11 she wins. If she throws a sum of 2, 3 or 12 ("craps") she loses. Otherwise call the sum she rolls the *point* P. The roller continually rolls until either (i) she rolls the sum P again or (ii) she rolls a 7. In the first case the roller wins the bet and in the second she loses. Spectators may bet with the roller or on a number of side bets listed on the table.

Our goal, or more correctly your goal, will be to compute the probability of winning the game of Craps. To tackle this problem, we can make one simplifying observation: Although a game of Craps entails many rolls (potentially), we can actually treat the game as a two stage branching game. What are the stages?

The first stage is, not surprisingly, the first roll of the dice. The possible outcomes of this roll will represent the first nodes in our tree. The events we should consider are Roll = 7 or11, Roll = 2,3 or 12, $P = 4$, $P = 5$, $P = 6$, $P = 8$, $P = 9$, $P = 10$. In the first event, we win, in the second we lose. In the other 6 we have rolled our point.

We now enter the second stage of the game. Here's the key idea for our analysis. While it may take a long while, we may be sure that on some roll you will either roll your point again and win or roll 7 or craps and lose the game. For simplicity, lets call the first event Roll Point and the second Roll 7. At each of the "point" nodes ($P = 4$ through $P = 10$) must compute a conditional probability. We can then complete the probability tree for Craps and determine the probability of winning the game.

Exercises for Section 5

Exercise 5.1. Compute the conditional probabilities

$$P(\text{Roll Point before } 7|\text{Point} = n)$$

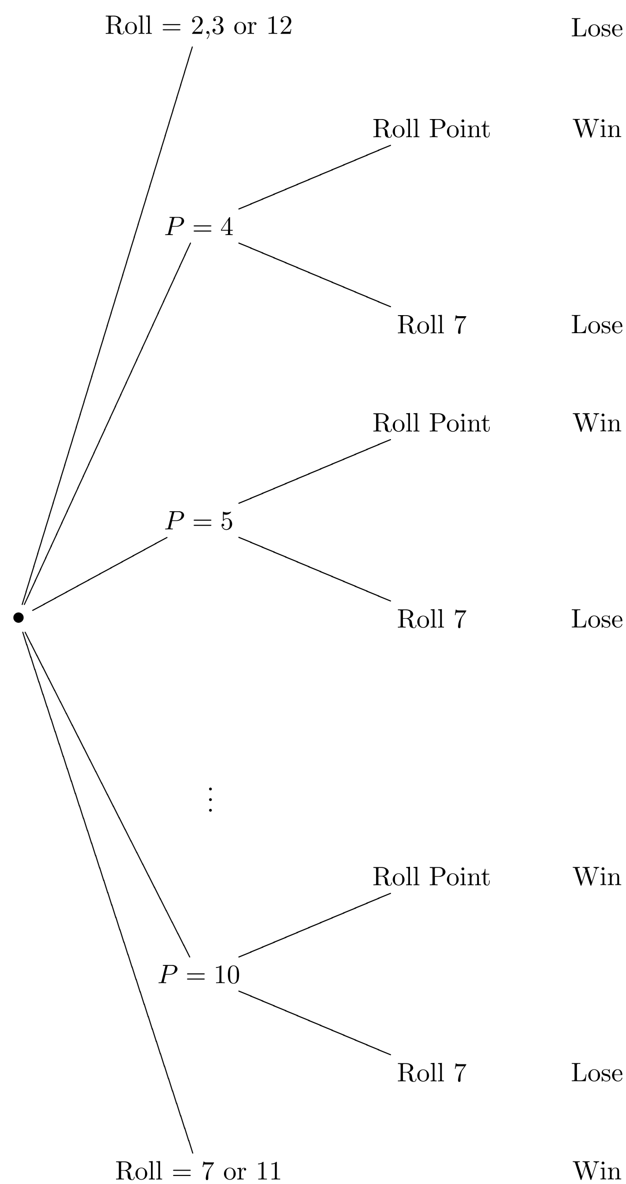
for $n = 4, 5, 6, 8, 9$ and 10.

Exercise 5.2. Write down the complete probability tree for the game Craps and compute the probability of winning the game. See Figure 3, below for a partial rendering of the final tree.

Exercise 5.3. In a casino, the "Don't Pass" bet wins if, on the first roll, a 2 or 3 is thrown and loses if, on the the first roll, a 7 is thrown. If a 12 is thrown on the first roll the bet is a "push" – no one wins. If neither a $2, 3, 7, 11$ nor 12 is rolled on the first roll then the first roll is called the "point". The "Don't Pass" bet wins if the shooter throws a 7 before the point and loses if the shooter throws the point before a 7. Compute the probability of winning a "Don't Pass" bet.

Exercise 5.4. You play a game of craps and win. What is the probability that you rolled a 6 on your first roll. Hint: Use Theorem 3.5 to compute $P(\text{Point} = 6|\text{Win})$.

Figure 3: Probability Tree for the Game Craps



6. Games of Pure Strategy

We now switch gears considerably. We consider games in which there is no element of chance and all depends on strategy. Furthermore, we now consider the role of the other player or players in the game. To accomodate multiple players (for our purposes two players), it is useful to introduce a value or *payoff* for each possible outcome of game. Of course, this value might differ between players for the same outcome. To illustrate the meaning of strategy and payoff, we consider two familiar examples of strategic games.

Consider the game of "Tic Tac Toe", a two-player game of pure strategy. Since both players presumably want to win, the issue of payoff is not a difficult one here. We can assign a value $+1$ for a Win, 0 for a tie and -1 for a loss for both players. What about strategy? As we know it helps to go first. Player 1 (or P_1 as we will write below) can make any of 9 choices. Player 2 (or P_2) can then make any one of 8 choices and the game continues. Strategy enters in right away. For example, while P_1's strategy on her first choice might be to play the middle or play a corner, P_2's strategy will be more complicated. Among the many choices, P_2's second choice strategy might be: play midddle if P_1 does not, otherwise play any available corner. Already, writing down a strategy for P_1's second choice is a bit complicated. (A tree is useful here!) But let's remember that we are not interested in playing "Tic Tac Toe," we are just interested in describing the game mathematically. We define a *strategy* S for P_1 to be a complete sequence of choices of moves in a game of "Tic Tac Toe" as first player. Similarly, a strategy T for P_2 will be a complete sequence of moves as second player. Thus the pair of strategies S and T will always produce an *outcome* which is either a Win for P_1 a Tie or a Win for P_2. Let's suppose that the outcome is a Win for P_1. With our assigned payoffs, we can express this as a table:

Player		P_2
	Strategy	T
P_1	S	$(1, -1)$

One simplification is immediate in this case. "Tic Tac Toe" is a *zero sum game.* Officially, we have:

Definition 6.1. A two player game of pure strategy is a *zero sum game* if, for each outcome, the payoff of that outcome for Player 1 is equal to negative the payoff of the outcome for Player 2.

Rather than write the pairs $(1, -1), (0, 0)$ or $(-1, 1)$ for the payoff of the outcome for P_1 and P_2 we may just write the payoff for P_1 with the assumption that the payoff for P_2 is just the negative of this value. Thus we rewrite our table as:

Player		P_2
	Strategy	T
P_1	S	1

Now imagine that we have a complete list $S_1, S_2, \ldots, S_M$ of all possible strategies for P_1 and a complete list $T_1, T_2, \ldots, T_N$ of all strategies for P_2. We then obtain a complete description of the game of "Tic Tac Toe" as an $M \times N$ table or *Payoff Matrix.*

(2) Payoff Matrix for "Tic Tac Toe"

Player			P_2		
	Strategy	T_1	T_2	$\ldots$	T_N
	S_1	1	0	$\ldots$	-1
P_1	S_2	-1	1	$\ldots$	0
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	S_N	1	0	$\ldots$	1

Of course, we have made up outcomes (i.e., payoffs of $0, \pm 1$) for some of the strategy pairs for the purposes of illustration.

We next consider a game which is not a zero-sum game to illustrate the importance of assigning payoffs to outcomes. We consider a simple auction, say of a baseball card, between two collectors. The first collector P_1 values the card at \$6 while the second collector values the card at \$7. For simplicity, lets assume that P_1 can bid either \$1, \$4 or \$10, call these strategies S_1, S_2, S_3. Assume P_2 can bid \$0, \$5 or \$9 and call these strategies T_1, T_2, T_3. The payoff of an outcome for a player is zero if the player loses the bidding and equals the difference between the player's bid and his valuation of the card if he wins the bidding. We thus obtain the following game matrix:

Payoff Matrix for the "Baseball Card Auction"

Player			P_2	
	Strategy	T_1	T_2	T_3
	S_1	$(5, 0)$	$(0, 2)$	$(0, -2)$
P_1	S_2	$(2, 0)$	$(0, 2)$	$(0, -2)$
	S_3	$(-4, 0)$	$(-4, 0)$	$(-4, 0)$

Once we have the payoff matrix for a given strategic game, we may begin to analyze how to play the game. For example, we might note that, in the "Baseball Card Auction", the payoff for P_1 of the outcome from playing strategy S_1 against T_1, T_2 and T_3 (namely 5, 0 ,0, respectively) are each at least as good as the corresponding payoffs for strategy S_2 (which are 2, 0, 0, respectively). Thus P_1 should never play S_2. This observation leads us to make the following:

Definition 6.2. A strategy S_1 for player P in a game of pure strategy *weakly dominates* a second strategy S_2 if the payoff for P of the outcome obtained by playing S_1 is greater than of equal to that of playing S_2 for all choices of strategy by the other players *and,* for at least one choice of strategies by the other players, the payoff of the outcome for P from playing S_1 is strictly greater than that from playing S_2. If the payoffs of the outcomes for P from playing S_1 are strictly greater than those of S_2 for all choices of the other players' strategies then we say S_1 *strongly dominates* S_2.

Using the notion of dominated strategies, we may give a complete analysis or *solution* to the "Baseball Card Auction". For observe that strategy S_1 weakly dominates strategy S_2 for P_1. This means that P_1 should not play S_2. Notice that we may not (yet!) conclude that P_1 should play S_1. Next observe that S_1 strongly dominates S_3 for P_1. Thus P_1 should not play S_3 either and we concluded that P_1 will play S_1. Given this information, it is now obvious how P_2 should play, namely she should play T_2 to maximize her payoff. We note one crucial feature of the outcome $(0, 2)$ corresponding to the strategy choices (S_1, T_2). If one player keeps their strategy fixed (e.g., P_1 stays with strategy S_1) then the other player can not improve her payoff by changing strategies. Succinctly put, neither player profits by *unilaterally* changing strategy. This leads us to make the important

Definition 6.3. An outcome of a strategic game is said to be a *Nash equilibrium* if no player can improve their payoff by unilaterally changing their strategy.

In the "Baseball Card Auction" we see that the outcome $(0, 2)$ corresponding to either the strategy pairs (S_1, T_2) or (S_2, T_2) are both Nash equilibria.

For zero-sum games, there is a direct procedure for locating Nash equilibria which we illustrate now.

Example 6.4. We consider a zero sum game in which each player has 3 strategies. We refer to this as a 3×3 zero sum game.

Payoff Matrix for a 3×3 Zero-Sum Game

Player			P_2	
	Strategy	T_1	T_2	T_3
	S_1	10	4	6
P_1	S_2	6	5	12
	S_3	2	3	7

Recall our convention for indicating payoffs wherein P_1's payoffs are given and P_2's payoffs are the negatives of these. Since P_1 would like the largest possible payoff, she would be smart to consider the worst case scenario for each of her strategies, that is the row minimum for each strategy S_1, S_2, S_3. Similarly, P_2 should look to see what is the worst case scenario for each of his strategies, i.e. the column maximum for each strategy column T_1, T_2, T_3. These are indicated in the following table.

Player			P_2		Row Minimum
	Strategy	T_1	T_2	T_3	
	S_1	10	4	6	4
P_1	S_2	6	5	12	5
	S_3	2	3	7	2
	Column Maximum	10	5	12	

Notice that P_1 can ensure herself an outcome of no less than \$5 by playing the strategy S_2. The strategy S_2 is called the *maximin* strategy for P_1 as it maximizes the worse case scenarios i.e. the row minima, for P_1. The strategy T_2 is called the *minimax* strategy as it minimizes the column maxima. Note that

$$\text{minimax} = \text{maximin} = 5.$$

The outcome 5 is called a *saddle point* of this game. We make all this official with the following definition.

Definition 6.5. Assume given a zero sum game with payoff matrix in our usual form, i.e. with postive entries corresponding to positive payoffs for Player 1 and losses for Player 2 and vice versa. The *maximin strategy* for Player 1 is the strategy (or row) such that the minimum payoff of outcomes which can occur with this strategy is at least as large as the minima for the other strategies available to Player 1. The *maximin* of the game is then defined to be the minimum payoff which occurs when Player 1 plays her maximin strategy. A *minimax strategy* for Player 2 is a strategy whose maximum payoff is no larger than the maximum payoff of other strategies for Player 2. The *minimax* of the game is the maximum payoff that occurs when Player 2 plays his minimax strategy. If maximin = minimax the game is said to have a *saddle point*. In this case the outcome corresponding this commona value is called the *saddle point*.

The connection between Nash equilibria and saddle points for zero sum games is contained in the following theorem.

Theorem 6.6. A saddle point for a zero sum game is a Nash equilibrium of the game. Conversely, a Nash equilibrium of a zero sum game must be a saddle point.

Proof. Let's start with the first statement. Suppose Player 1 is playing a maximin stategy, say S, and Player 2 is playing a minimax strategy, say T. Let

$$m = \text{minimax} = \text{maximin}$$

so that m is the payoff corresponding to the strategy pair S, T. Now suppose Player 1 switches to S' but Player 2 stays with strategy T (this is a unilateral change of strategy for P_1). Since m corresponds, in particular, to the minimax, the value m is the maximum payoff for the column corresponding to the strategy T. Thus the payoff for the pair S', T is smaller than that for S, T and Player 1 cannot unilaterally improve her outcome. A similar argument shows Player 2 cannot unilaterally improve his outcome either and so the saddle point is a Nash equilibrium by definition.

For the converse, suppose the strategy pair S, T yields an outcome m which is a Nash equilibrium. We then see that m is the maximum for the column corresponding to T. For otherwise, Player 1 could unilaterally change strategies and improve her payoff. Thus $m \geq$ minimax, since the latter is the smallest column maximum. Similar reasoning shows $m \leq$ maximin. But now we observe that the inequality maximin $\leq$ minimax always holds. (See Exercise 6.7.) Thus

$$m = \text{minimax} = \text{maximin}$$

is a saddle point, as needed. □

Exercises for Section 6

Exercise 6.1. It is a theorem that the first player in "Tic Tac Toe" can always ensure at least a tie. Explain precisely what this theorem asserts about the payoff matrix for "Tic Tac Toe" described above. Can you make a related conjecture about Player 2?

Exercise 6.2. Find the maximin and minimax stragies and payoffs for the following zero sum game payoff matrices. Decide in each case if there is a saddle point.

(a)

Player			P_2	
	Strategy	T_1	T_2	T_3
	S_1	5	-1	6
P_1	S_2	4	15	22
	S_3	-8	-3	6

(b)

Player			P_2	
	Strategy	T_1	T_2	T_3
	S_1	-2	-1	6
P_1	S_2	5	-5	1
	S_3	-6	2	-3

(c)

Player			P_2	
	Strategy	T_1	T_2	T_3
	S_1	12	8	7
P_1	S_2	14	-8	-3
	S_3	6	2	5

Exercise 6.3. Write down a payoff matrix with all entries positive for a 4×4 zero sum game satisfying:
(a) at least one strategy for P_1 weakly dominates another strategy and at least one strategy for P_2 weakly dominates another
(b) there is a saddle point

Exercise 6.4. Consider the following variation on the "Baseball Card Auction" above. Here P_1 values the card at \$7 and can bid \$2, \$4, \$6 or \$8 while P_2 values the card at 8 and can bid \$1, \$3, \$7, or \$9. Write down the payoff matrix for this game. Identify all weakly dominant pairs of strategies for both players and find all Nash equilibria.

Exercise 6.5. Consider the following variation on the "Dollar Auction". Two players bid simultaneously for a dollar. The high bidder gets the dollar but must pay their bid. The low bidder must pay their bid and gets nothing (thus has a zero or negative payoff). P_1 can only bid the amounts \$.50, \$.95, \$1.10, or \$1.20 and P_2 can only bid the amounts \$.40, \$.80, \$1.00 or \$1.50. Write down the payoff matrix for this game. Identify all weakly dominant pairs of strategies for both players and find all Nash equilibria.

Exercise 6.6. Consider another variation on the "Dollar Auction". Players 1 and 2 bid simultaneously. As usual, the high bidder gets the dollar but must pay their bid. The low bidder must pay their bid and gets nothing. Player 1 can bid only multiples of even amounts and Player 2 can only bid odd amounts. Is there a Nash equilibrium for this game. Justify your answer carefully.

Exercise 6.7. Justify the statement that for every payoff matrix,

$$\text{maximin} \leq \text{minimax}.$$

7. Mixed Strategy Equilibria: The MiniMax Theorem

Most zero sum games do not have a saddle point. In this section, we discuss how to play such zero sum games. Our analysis will lead us to a formulation of Von Neumann's MiniMax Theorem.

We begin with a simple game example, a two player, two strategy game game, known as "Matching Pennies". Each player has a coin and thus two strategies, to play H = Heads or T = Tails. Player 1 gets $1 from Player 2 if the coins match. Player 2 gets $1 from Player 1 if the coins do not match. The payoff matrix is as shown:

Payoff Matrix for "Matching Pennies"

Player		P_2	
	Strategy	H	T
	H	1	-1
P_1	T	-1	1

Clearly, if we play this game one time there is no preferred strategy. Both players have a 50% of winning with either strategy. But suppose we agree to play the game many (say, 100) times. We may then ask:

Question 1: What fraction p of the time should Player 1 play Heads?

and

Question 2: What fraction q of the time should Player 2 play Heads?

Although a step in the right direction, answering these questions is not sufficent to determine the net payoff or loss to each player after the 100 rounds. For example, if Player 1 decides to play Heads 75 times and Tails 25 times (i.e chooses $p = .75$) and Player 2 decides to play Heads 50 times and Tails 50 times (i.e. chooses $q = .5$) the actual net payoff or loss depends on the actual order in which each player plays Heads and Tails. We can, however, determine the *expected value* $\mathrm{E}(p,q)$ of the payoff for these choices of p and q. Here $\mathrm{E}(p,q)$ will be the payoff (or loss) for Player 1 and $-\mathrm{E}(p,q)$ will be the payoff (or loss) for Player 2. To compute the value of $\mathrm{E}(p,q)$ use the following table:

Player		P_2		
	Probablility		q	$1-q$
		Strategy	H	T
P_1	p	H	1	-1
	$1-p$	T	-1	1

The events "Player 1 Chooses Heads" and "Player 2 Chooses Heads" are independent. Thus, by the Law of Independent Events, the probability of both players choosing heads is $P(HH) = pq$. Using this same argument for all possible combinations gives $P(HT) = p(1-q), P(TH) = (1-p)q$ and $P(TT) = (1-p)(1-q)$.

Thus the expected value is given by the formula

$$\begin{aligned} \mathrm{E}(p,q) &= 1\cdot P(HH) - 1\cdot P(HT) - 1\cdot P(TH) + 1\cdot P(TT) \\ &= pq - (1-q)p - (1-p)q + (1-p)(1-q) \\ &= 4pq - 2(p+q) + 1 \\ &= (2p-1)(2q-1) \end{aligned}$$

where in the last step we have done a bit of algebra. It is worth noting that what we have produced may be viewed as a new zero sum game: Players 1 and 2 pick numbers, p and q respectively, between 0 and 1 inclusive. The set of numbers between zero and one inclusive is written in interval notation as $[0,1]$. The payoff is then $E(p,q)$. We call this game the *Expected Value Version* of the original game. We may visualize the payoff matrix very simply as

Payoff Matrix for Expected Value Version of "Matching Pennies"

Player		P_2
	Strategy	q
P_1	p	$\mathrm{E}(p,q) = (2p-1)(2q-1)$

A moment's thought reveals the Nash equilibrium for this new game, namely P_1 plays $p^* = .5$ and P_2 plays $q^* = .5$ and the payoff is zero. For notice that if either player plays .5, from the formula for $\mathrm{E}(p,q)$ we see that the other player cannot improve, indeed cannot change the payoff from zero by changing his or her strategy. That we should play Heads and Tails equally often certainly appeals to common sense! The probabilities $p^* = .5$ and $q^* = .5$ are said to yield the *mixed strategy equilibrium* of this zero sum game. The value

$$E(p^*, q^*) = 0$$

is called the *solution* to the game. We make these definitions for general zero sum games, but first make sum remarks about notation.

By an $M \times N$ *Zero Sum Game* we mean a zero sum game in which Player 1 has M strategies, say, $S_1, S_2, \ldots, S_M$ and Player 2 has N strategies, say, $T_1, T_2, \ldots, T_N$. In the Expected Value version of the game, Player 1 will be choosing probabilities $p_1, p_2, \ldots, p_M$ for playing the strategies $S_1, S_2, \ldots, S_M$. The p_i are then elements of the interval $[0,1]$. Moreover, we must have

$$p_1 + \cdots p_2 + \cdots + p_M = 1.$$

Note that this condition implies that

$$p_M = 1 - (p_1 + p_2 + \cdots + p_{M-1}).$$

It follows that choosing M probabilities $p_1, p_2, \ldots, p_M$ is equivalent to picking $M-1$ probabilities $p_1, p_2, \ldots, p_{M-1}$ with

$$p_1 + \cdots p_2 + \cdots + p_{M-1} \le 1$$

and setting

$$p_M = 1 - (p_1 + p_2 + \cdots + p_{M-1}).$$

Above we have used the latter representation when we take $p_1 = p$ and $p_2 = 1 - p$. Sometimes it will be more efficient to use the former as in the following definitions.

Definition 7.1. Suppose given a zero sum game in which Player 1 has strategies $S_1, S_2, \ldots, S_M$ and Player 2 has strategies $T_1, T_2, \ldots, T_N$. Consider the Expected Value Version of the game in which Player 1 chooses probabilities $p_1, p_2, \ldots, p_M$ summing to 1 and Player 2 chooses probabilities $q_1, q_2, \ldots, q_N$ summing to 1 yielding the expected value $\mathrm{E}(p_1, p_2, \ldots, p_M, q_1, q_2, \ldots, q_N)$. A *mixed strategy equilibrium* is a choice of probabilities $p_1^*, p_2^*, \ldots, p_M^*$ summing to 1 for Player 1 and probabilites $q_1^*, q_2^*, \ldots, q_N^*$ summing to 1 for Player 2 such that these strategies yield a Nash equilibrium of the corresponding Expected Value game. The expected value $\mathrm{E}(p_1^*, p_2^*, \ldots, p_M^*, q_1^*, q_2^*, \ldots, q_N^*)$ is called the *solution* of the game. If some $p_j^* = 1$ and all other $p_i^* = 0$ and, similarly, some $q_k^* = 1$ while all other $q_i^* = 0$ then the game is said to have a *pure strategy equilibrium* given by the strategies S_j, T_k. The payoff of the outcome corresponding to these strategies is then the solution of the game.

We can now state the MiniMax Theorem originally due to John Von Neumann.

Theorem 7.2. *(MiniMax Theorem for Zero Sum Games)* Every zero sum game has a mixed strategy equilibrium and a solution. □

Unfortunately, the Minimax Theorem does not tell us how to compute the mixed strategy equilibrium probabilities and the solution of a zero sum game. For games with many strategies, this can be a difficult computational problem. Fortunately, for the case of 2×2 zero sum games, the mixed strategy equilibrium probabilities and the solution can be easily computed, as we now explain.

We focus on a concrete example.

Example 7.3. Consider the payoff matrix for a 2×2 zero sum game below:

Player		P_2	
	Strategy	X	Y
	A	4	3
P_1	B	2	7

Here, for convenience, we have denoted the strategies for P_1 by A, B and for P_2 by X, Y. Following the procedure for "Matching Pennies", we compute the expected value $\mathrm{E}(p, q)$ when P_1 plays A with probability p and P_2 plays X with probability q. We have

$$\mathrm{E}(p, q) = 4pq + 3(1 - q)p + 2(1 - p)q + 7(1 - p)(1 - q).$$

Now we might be able to perform some clever algebra, to find the mixed strategy probabilities p^* and q^*. Rather than pursuing this, we take a different tack. We focus on Player 1 and imagine, for a moment, that Player 2 plays strategy X everytime. What is the expected value of the payoff for Player 1 if she continues to play A with probability p and B with probability $1 - p$? Let's write this expected value as $\mathrm{E}_X(p)$. Since, with Player 2 fixed on strategy X, the payoffs for Player 1 are 4 for playing A and 2 for playing B we see

$$\mathrm{E}_X(p) = 4p + 2(1 - p).$$

Next let's imagine that Player 2 plays Y every time. Then the expected value $\mathrm{E}_Y(p)$ for Player 1 from her probability choice p is given by

$$\mathrm{E}_Y(p) = 3p + 7(1-p).$$

We now do something which seems a bit arbitrary. We set

$$\mathrm{E}_X(p) = \mathrm{E}_Y(p)$$

and solve for the solution p^*. This yields

$$4p^* + 2(1-p^*) = 3p^* + 7(1-p^*) \Longrightarrow p^* = 5/6.$$

To find q^*, we carry out the same procedure from Player 2's point of view. Here

$$\mathrm{E}_A(q) = -4q - 3(1-q)$$

is the expected value of the payoff for Player 2 when he chooses probabilty q and Player 1 plays only A. Similarly,

$$\mathrm{E}_B(q) = -2q - 7(1-q)$$

is the expected value of the payoff for Player 2 when he chooses probabilty q and Player 1 plays only B. Equating and solving gives

$$-4q^* - 3(1-q^*) = -2q^* - 7(1-q^*) \Longrightarrow q^* = 2/3.$$

We now claim two facts:

(1) The pair of probabilities $p^* = 5/6$ and $q^* = 2/3$ are the mixed strategy equilibrium of the "Random 2×2 Zero Sum Game".

(2) The values

$$E_X(p^*) = E_Y(p^*) = -E_A(q^*) = -E_B(q^*)$$

and their common value, 22/6 is the solution of the game.

These facts are a consequence of our next theorem and our analysis of this example is complete.

Theorem 7.4. Suppose given a 2×2 zero sum game in which Player 1 has strategies A and B and Player 2 has strategies X and Y. Let $\mathrm{E}_X(p)$ and $\mathrm{E}_Y(p)$ be the expected payoffs for Player 1 when Player 2 plays only X and only Y, respectively. Assume we can solve the equation $\mathrm{E}_X(p^*) = \mathrm{E}_Y(p^*)$ for $p^* \in [0,1]$. Similarly, let $\mathrm{E}_A(p)$ and $\mathrm{E}_B(p)$ be the expected payoffs for Player 2 when Player 1 plays X only and and Y only, respectively. Assume we can solve the equation $E_A(q^*) = E_B(q^*)$ for $q^* \in [0,1]$. Then the pair of probabilities p^* and q^* are a mixed strategy equilibrium of the game. Further, we have

$$E_X(p^*) = E_Y(p^*) = -E_A(q^*) = -E_B(q^*)$$

and their common value is a solution of the game.

Proof. The proof is based on the following equations which can be easily verified for any 2×2 zero sum game payoff matrix:

$$\text{Equation 1:} \qquad \mathrm{E}(p,q) = q\mathrm{E}_X(p) + (1-q)\mathrm{E}_Y(q)$$

and

$$\text{Equation 2:} \qquad \mathrm{E}(p,q) = -p\mathrm{E}_A(q) - (1-p)\mathrm{E}_B(q)$$

The first equation simply reflects the fact that Player 2 plays X with probability q and strategy Y with probability $1-q$. Similarly, the second equation is a direct consequence of the fact that Player 1 plays A with probability p and strategy B with probability $1-p$. But now suppose that Player 1 plays $p = p^*$. Then, by our choice of p^* we have $E_X(p^*) = E_Y(p^*)$ and so from the first equation we see

$$\begin{aligned} \mathrm{E}(p^*, q) &= q\mathrm{E}_X(p^*) + (1-q)\mathrm{E}_Y(p^*) \\ &= q\mathrm{E}_X(p^*) + (1-q)\mathrm{E}_X(p^*) \\ &= \mathrm{E}_X(p^*). \end{aligned} \tag{3}$$

This means that no matter what value of q Player 2 chooses the outcome, when Player 1 stays with $p = p^*$, remains unchanged. Similarly, when Player 2 plays $q = q^*$ then, for any choice of probabilty p by Player 1, we have

$$\begin{aligned} \mathrm{E}(p, q^*) &= -p\mathrm{E}_A(q^*) - (1-p)\mathrm{E}_B(q^*) \\ &= -p\mathrm{E}_A(p^*) - (1-p)\mathrm{E}_A(q^*) \\ &= -\mathrm{E}_A(q^*). \end{aligned} \tag{4}$$

It follows directly that $p = p^*$ and $q = q^*$ yield a Nash equilibrium for the Expected Value version of our zero-sum game. Finally notice taking $q = q^*$ in (3) we have

$$E(p^*, q^*) = E_X(p^*) = E_Y(p^*)$$

while takine $p = p^*$ in (4) gives

$$E(p^*, q^*) = -E_A(q^*) = -E_B(q^*).$$

Thus

$$E_X(p^*) = E_Y(p^*) = -E_A(q^*) = -E_B(q^*)$$

and all agree with the solution $E(p^*, q^*)$ of the game. □

With Theorem 7.4, we seem to have proven that every 2×2 zero sum game has a solution. However, our proof makes essential use of the assumption that both equations $E_X(p^*) = E_Y(p^*)$ and $E_A(q^*) = E_B(q^*)$ can be solved for p^* and q^* in $[0, 1]$, respectively. What happens when one or both of these equations has no solutions? As usual, we consider an example:

Example 7.5. We describe a situation where one of the equations $E_X(p^*) = E_Y(p^*)$ and $E_A(q^*) = E_B(q^*)$ cannot be solved.

Payoff Matrix for 2×2 Zero Sum Game

Player		P_2	
	Strategy	X	Y
	A	2	3
P_1	B	10	6

Observe that

$$E_X(p^*) = E_Y(p^*) \Longrightarrow 2p^* + 10(1-p^*) = 3p^* + 6(1-p^*) \Longrightarrow p^* = 4/5.$$

On the other hand, we have

$$E_A(q^*) = E_B(q^*) \Longrightarrow -2q^* - 3(1-q^*) = 10q^* + 6(1-q^*) \Longrightarrow q^* = -3/5.$$

We conclude that that $E_A(q^*) = E_B(q^*)$ has no solution for q^* between zero and one. We can't apply Theorem 7.4. Taking a closer look at the payoff matrix, however, we see that, for P_1, the strategy B strictly dominates strategy A. Thus P_1 should play B always and so we should set $p^* = 0$. Now, with P_1 playing B always, it's just common sense for P_2 to play Y always i.e. to take $q^* = 0$. Thus we conclude that $p^* = 0, q^* = 0$ give the *pure* strategy equilibrium A, X and the pure strategy solution 6. Alternately, we could observe the payoff matrix above has a saddle point at 6 and so this outcome is the solution (see Exercises 7.3 and 7.4 below.) We have thus solved the game.

We conclude by remarking that the ideas above combine to give a complete proof of the MiniMax Theorem in the special case of 2×2 zero sum games (see Exercise 7.5). In particular, we are now able to solve any 2×2 zero sum game using either the method of Example 7.3 or that of Example 7.5.

Exercises for Section 7

Exercise 7.1. Find the mixed strategy equilibrium and solution for the following 2×2 zero sum games.

(a)

Player		P_2	
	Strategy	X	Y
	A	-5	3
P_1	B	4	-1

(b)

Player		P_2	
	Strategy	X	Y
	A	-2	10
P_1	B	4	6

(c)

Player		P_2	
	Strategy	X	Y
	A	3	-3
P_1	B	2	1000

Exercise 7.2. We view the battle between a Batter and Pitcher in a baseball game as 2×2 zero sum game. The Batter can guess f= Fastball or c=- Curveball and swing accordingly. The Pitcher can throw F= FastBall or C = Curveball. When the batter guesses Fastball and the pticher throws Fastball, the batter is hitting .400. When the batter guesses Fastball and the pticher throws Curveball, the batter is hitting .150. When the batter guesses Curveball and the pticher throws Curveball, the batter is hitting .360. When the batter guesses Curveball and the

pticher throws Fastball, the batter is hitting .220. Write down the payoff matrix for this game and find the mixed strategy equilibrium and the solution.

Exercise 7.3. Suppose we are given a 2×2 zero-sum game with payoff matrix

Player		P_2	
	Strategy	X	Y
	A	a	b
P_1	B	c	d

such that $E_X(p^*) = E_Y(p^*)$ has no solution. Prove that the payoff matrix has a saddle point.
Hint. Note that either $E_X(p) < E_Y(p)$ for all $0 \leq p \leq 1$ or $E_X(p) > E_Y(p)$ for all $0 \leq p \leq 1$. Show that either way Player 2 has a weakly dominany strategy. Then show this forces a saddle point.

Exercise 7.4. Suppose we are given a 3×3 zero sum game in which P_1 has strategies S_1, S_2, S_3 and P_2 has strategies T_1, T_2, T_3. Suppose the strategies S_1 and T_1 yield a saddle point, say m, of the payoff matrix. Prove the game has S_1, T_1 as a pure strategy equilibrium and so solution m. Your proof should make use of the definition of the saddle point and thus the notions of minimax and maximin. Does your proof generalize to any $n \times n$ zero sum game?

Exercise 7.5. Show that Exercises 7.3 and 7.4 combined with Theorem 7.4 give a complete proof of the MiniMax Theorem (Theorem 7.2) in the 2×2 case.

8. Partial Conflict Games: Social Dilemmas

Many strategic games that occur in political or social situations are not zero-sum or *total conflict* games but rather admit of some cooperation among the players. Such games are called *partial conflict* games. The analysis of these games has a very different flavor from zero-sum games. The most interesting problems that emerge in this area actually live outside the realm of mathematics. As we will see, the very simple mathematical formulation of a 2×2 Partial Conflict Game pinpoints some extremely difficult issues arising in social and political conflict theory.

Prisoner's Dilemma. The most famous example 2×2 Partial Conflict Game is the Prisoner's Dilemma. We use this example to motivate our discussion. The set-up can be described in many ways but the classic version features two prisoners accused of a common crime held in separate prison cells. Each prisoner has a choice between two options: to rat on their partner or or to stay silent. If both prisoners rat each other out then both get 10 year sentences. If both prisoners remain silent then both receive 1 year sentences. Finally, if one prisoner rats and the other stays silent the ratting prisoner goes free and the silent prisoner gets 20 years. How would you play this game? Notice that the psychology of your opponent is, all of a sudden, of extreme importance? What (s)he will do is the basic question you must resolve.

We write down the payoff matrix for this two player game. Since this is not a zero-sum game, we must use ordered pairs (row player payoff, column player payoff) again. We will treat a prison sentence as a negative payoff. Thus, a 20 year term will be a payoff of -20. Also, we label the ratting out strategy "Defection" and the stay silent strategy "Cooperate". We then have the payoff matrix.

Prisoner's Dilemma

	Cooperate	Defect
Cooperate	$(-1,-1)$	$(-20,0)$
Defect	$(0,-20)$	$(-10,-10)$

The significance of the Prisoner's Dilemma lies in the following two observations: First, mutual defection yields a Nash equilibrium by definition. Second, mutual cooperation yields a better outcome for *both* players. Furthermore, "Defect" is a dominant strategy for both players and yet a better outcome for both is obtained by choosing to "Cooperate". Much effort has been made to reconcile this basic conflict in social and political science: to encourage cooperation of parties in the face of a Prisoner's Dilemma.

In the study of Partial Conflict games, it is helpful to standardize the payoffs. Notice that there are generally four payoffs for each player which we may order 1, 2, 3 and 4 with, as always, the larger the payoff the better. We then obtain what is called the *normal form* for the Prisoner's Dilemma:

Prisoner's Dilemma in Normal Form

	C	D
C	$(3,3)$	$(1,4)$
D	$(4,1)$	$(2,2)$

Chicken. The next game we consider may be motivated as an extremely stupid test of wills. Two teenagers drive their cars toward each other on a narrow road. Each driver has two options: to "Swerve" and thus avoid a collision or to "Not Swerve" and drive straight. The consequences are obvious. If one player swerves and the other does not, the swerver is embarrassed while the non-swerver is triumphant. If neither player swerves then the worst case scenario occurs for both, a head-on collision. Finally, if both players swerve they are both a bit embarrassed but neither has been shown up. We let "Swerve" correspond to "Cooperate" or "C" and "Not Swerve" correspond to "Defect" or "D". We then have the following normal form payoff matrix:

Chicken in Normal Form

	C	D
C	$(3,3)$	$(2,4)$
D	$(4,2)$	$(1,1)$

Here we notice that there are two Nash equilibriums occurring at (C, D) and (D, C). What is the dilemma? Think about your strategy for playing this game. If you plan to defect, you must be assured your opponent will cooperate, otherwise you will meet certain disaster. Said differently, to succeed with the strategy of defection in Chicken you must convince your opponent that you are fully capable of defecting. Even better, your opponent would be convinced that you *will* defect. This is sometimes call the "Madman" strategy. The question of how and whether the U.S. should employ this strategy for "playing" the nuclear game of chicken with the Soviet Union was an extremely controversial question during the Cold War.

We next observe that the notion of a Prisoner's Dilemma can be extended to more than two players and more than two strategies. The definition requires use of the notions of weakly dominated strategies. It also requires some notational choices. We will suppose that we have n players $P_1, \ldots, P_n$. Each player can choose strategies from a set S. We write $(s_1, \ldots, s_n)$ for a strategy choice by each player. That is $(s_1, \ldots, s_n)$ means P_1 has chosen strategy s_1, P_2 has chosen strategy s_2, etc. Of course, a strategy choice by each player yields an outcome or payoff for each player. Recall that a strategy choice s_i for P_i *weakly dominates* another choice t_i if the payoff for playing s_i is always at least as good as playing t_i for P_i against any choices by the other players with s_i giving a better outcome in at least one case. We next extend this notion to a sequence of strategy choices $(s_1, \ldots, s_n)$ for all players.

Definition 8.1. We say a sequence strategy choice $(s_1, \ldots, s_n)$ *weakly dominates* the strategy choice $(t_1, \ldots, t_n)$ if the payoff for each player from the choice $(s_1, \ldots, s_n)$ is at least as high as the payoff from $(t_1, \ldots, t_n)$ with at least one player doing better.

Recall now that, in technical terms, the Prisoner's Dilemma is vexing because if each player chooses their dominant strategy of defection the outcome is worse for all then if they had all chosen to cooperate. We generalize this to the n-player game as follows.

Definition 8.2. We say an n player game is a *generalized Prisoner's Dilemma* if there exist two sequences of strategy choice

$$(c_1, \ldots, c_n) \text{ and } (d_1, \ldots, d_n)$$

such that

(1) $(c_1, \ldots, c_n)$ weakly dominates $(d_1, \ldots, d_n)$ and

(2) For each player P_i, strategy d_i weakly dominates every other strategy choice for P_i.

Vickrey Auctions. We illustrate the notion of a generalized Prisoner's Dilemma with a famous example from the theory of auctions. A Vickrey auction is a sealed bid auction in which the object goes to the player with the highest bid. However, unlike a standard auction the price paid for the object is the *second highest bid*. Thus if everyone bids \$100 but you bid \$120 you win the object and pay \$100 for the object. Suppose now that there are n-players and each player values the object for sale at a price v_i. The strategies here are numbers ≥ 0. We first observe that the strategy v_i weakly dominates every other strategy for P_i. To see this, we let u_i be another number and fix strategies s_j for all other players P_j. We must argue that the payoff to P_i from $(s_1, \ldots, s_{i-1}, v_i, s_{i+1} \ldots, s_n)$ is at least as good for P_i as the payoff from $(s_1, \dots, s_{i-1}, u_i, s_{i+1} \dots, s_n)$. There are two cases to consider each with some subcases:

Case 1 Assume $u_i > v_i$. First suppose that P_i wins the object with the bid v_i. Then she also wins the bid with u_i and pays the same either way (namely the second highest bid). Next suppose P_i loses with the bid v_i. If she also loses with u_i then the outcome is the same. So assume the bid u_i wins but v_i loses. Then there is some s_j satisfying $u_i > s_j > v_j$. But this means that P_i must pay at least s_j for the object which is more than her valuation (which is v_i). Thus playing strategy v_i is strictly better in this case.

Case 2 Assume $v_i > u_i$. Suppose that P_i wins the object with the bid u_i. The again she also wins the bid with u_i and pays the same either way. If P_i loses with the bid u_i but wins with v_i then she gets the object for less than her valuation and so gains a positive payoff. Thus playing strategy v_i is strictly better in this case.

To complete our argument, we set

$$(c_1, \dots, c_n) = (v_1, \dots, v_n) \text{ and } (d_1, \dots, d_n) = \left(\frac{v_1}{2}, \dots, \frac{v_n}{2}\right).$$

We have $c_i = v_i$ satisfies (2) as in Definition 8.2. Thus it remains to observe that the strategy sequence $(d_1, \dots, d_n)$ weakly dominates $(c_1, \dots, c_n)$. But this is immediate since the winning player remains the same with either strategy sequence but pays half as much with the sequence $(d_1, \dots, d_n)$ as with the sequence $(c_1, \dots, c_n)$

We leave some further examples of Social Dilemmas and their properties to the exercises:

Exercises for Section 8

Exercise 8.1. Battle of the Sexes. In this game, a husband and a wife are deciding what to do on a Saturday night. The wife would prefer to go to a concert and the husband to a play. Both would prefer to go to either event together as opposed to splitting up. Each spouse has two options: to pick "Concert" or "Play". If both pick the same they go to that event. Otherwise they go alone to their preference. Identify one of these strategies as cooperation and the other as defection for each player and determine the normal form payoff matrix. Find all Nash Equilibria.

Battle of the Sexes

	C	D
C	(,)	(,)
D	(,)	(,)

Exercise 8.2. In the 2008 Democratic Primary, the opponents Barak Obama and Hillary Clinton had to choose between two basic strategies: going negative and staying positive. Define cooperation to be staying positive and defection as going negative. Explain in a paragraph or two how this may be represented as a 2×2 Partial Conflict Game. Justify your payoff choices carefully. Put the game in normal form and determine all Nash equilibria.

Exercise 8.3. Create a real-life example of a two player partial conflict game. Carefully explain each player's strategies and justify the various payoffs. Put your game in normal form and find all Nash Equilibria.

Exercise 8.4. Show that a *collective action problem* can lead to a generalized Prisoner's Dilemma. define your game and strategies and carefully prove that you have a generalized Prisoner's Dilemma using the definition.

9. Nash's Equilibrium Theorem

The preceding sections have shown that Nash Equilibria should not be expected in typical games. Indeed, in zero-sum games Nash Equilibria correspond to saddle points which are, relatively speaking, quite rare. Nevertheless, in what has become the seminal result in mathematical game theory John Nash proved the existence of Nash Equilibria in every game of a certain sort. Nash's Theorem implies Von Neumann's MiniMax Theorem and is, perhaps, best understood in this context.

To appreciate the depth Von Neumann's MiniMax Theorem (Theorem 7.2), we need only imagine trying to solve a 3×3 zero sum game, e.g.

Payoff Matrix for a 3×3 Zero Sum Game

Player			P_2	
	Strategy	X	Y	Z
	A	-4	3	2
P_1	B	8	2	5
	C	0	8	-1

We would have to now introduce probabilites p_1, p_2 with $p_1 + p_2 \leq 1$ and $p_3 = 1 - p_1 - p_2$ for Player 1 and q_1, q_2 with $q_1 + q_2 \leq 1$ and $q_3 = 1 - q_1 - q_2$. We could then write down the expected value for the game payoff as a real-valued function $\text{E}(p_1, p_2, q_1, q_2)$ of these four variables. Finding the solution means finding a Nash equilibrium of the Expected Value version of the game which looks like this:

Payoff Matrix for Expected Value Version of a 3×3 Zero Sum Game

Player		P_2
	Strategy	q_1, q_2 with $q_1, q_2 \geq 0$, $q_1 + q_2 \leq 1$
P_1	p_1, p_2 with $p_1, p_2 \geq 0$, $p_1 + p_2 \leq 1$	$\text{E}(p_1, p_2, q_1, q_2)$

In some cases, we can eliminate dominated strategies from the original payoff matrix to reduce the 3×3 payoff matrix to a 2×2 matrix and then find the solution. Unfortunately, this method won't work with the game above. (However, see Exercise 9.1 for an example where this ad hoc approach does work.) Solving $n \times n$ zero sum games for $n \geq 3$ in general requires advanced methods from the area of mathematics called *linear programming*. We will not enter into this theory here.

Instead we discuss a remarkable theorem proved by John Nash in the 1950s which includes the MinMax Theorem as a special case. Nash considers games in which the strategy choices are not finite sets like $S = \{S_1, S_2, \ldots, S_M\}$ but infinite sets like the interval $[0, 1]$ or the two-dimensional set of pairs $\{(p_1, p_2) | p_1, p_2 \geq 0 \text{ and } p_1 + p_2 \leq 1\}$ from the above Expected Value game. We write S for the set of strategies for P_1 and T for the set of strategies for P_2. (We consider only two players for convenience, although Nash's Theorem does not require this restriction.) A *payoff funtion* $P(s, t)$ for these strategy sets is just a pair real valued functions $p_1(s, t)$ and $p_2(s, t)$ for s, t elements of S and T respectively. The number $p_1(s, t)$ is the payoff of the strategy choices s, t for Player 1 and $p_2(s, t)$ the payoff for Player 2. Note that the game is zero sum if $p_1(s, t) = -p_2(s, t)$ for all strategy pairs but we don't require this either. We can summarize our game with the following table:

A 2×2 Game with Infinite Strategy Sets

Player		P_2
	Strategy	$t \in S_2$
P_1	$s \in S_1$	$P(s, t) = (p_1(s, t), p_2(s, t))$

To illustrate

Example 9.1. We consider another variation on the Baseball Card Auction. In this case, P_1 can bid any dollar amount s with $0 \leq s \leq 100$ and P_2 can bid any dollar amount with t with $10 \leq t \leq 90$. For convenience, we assume that if both players bid the same amount then Player 1 wins the bidding. Also we use the standard auction rule: If a player loses the bidding his or her payoff is zero. Assuming P_1 values the card at \$75 dollars and P_2 at \$60, the payoff functions are seen to be:

$$p_1(s, t) = \begin{cases} 75 - s & \text{for} \quad s \geq t \\ 0 & \text{for} \quad s < t \end{cases}$$

and

$$p_2(s, t) = \begin{cases} 60 - t & \text{for} \quad t > s \\ 0 & \text{for} \quad s \leq t \end{cases}$$

Nash's Theorem below requires some technical hypotheses on the payoff functions $p_1(s, t)$ and $p_2(s, t)$ and the strategy sets S and T. We discuss these informally here. First, Nash requires both the payoff functions to be *continuous.* (Actually, Nash requires a weaker but more technical condition but we will content ourselves with making this requirement.) For real-valued functions of one variable, continuity means the graph of the function can be drawn without lifting the pencil. Continuity of functions of several variables involves a fair amount of technicality from calculus (namely, the limit) and so we omit a formal definition. We remark that the payoff functions above are *not* continuous functions.

Nash's Theorem also requires strong hypotheses on the strategy sets. Specifically, the theorem requires the strategy sets S and T to be *closed, bounded* and *convex.* We give informal definitions of these notions.

First we say a subset S in n-dimensional space is *closed* if S contains all the points on it's boundary. Thus an interval $[a, b]$ is closed on the real line but an interval (a, b) which excludes the endpoints is not closed since the boundary points a and b are not contained in the interval. (Accordingly, the interval $[a, b]$ is called a *closed* interval and the interval(a, b) an *open* interval.) In the plane (two-dimensional space), a circular or rectangular region is closed if and only if the region contains the boundary circle or rectangle.

Next we say a subset S of n-dimensional space is *bounded* if there is some large distance D such that every pair of points in S are distance $\leq D$ apart. Thus intervals are bounded on the real line and rectangular regions are bounded in the plane. The set of nonnegative real numbers is not bounded nor is the set of all points in the plane with postive x and y coordinates.

Finally, we say a subset S of n-dimensional space is *convex* if the line segment joining every pair of points in S is wholly contained in S. Convex sets on the real line are simply the intervals (open or closed). In the plane, a disc is convex as is a rectangluar region, but a triangular region need not be. It is not difficult to check that sets like

$$S = \{(p_1, p_2, \dots, p_M) | p_i \geq 0 \text{ and } p_1 + p_2 \cdots + p_M = 1\}$$

arising in the Expected Value version of a zero sum game are closed, bounded and convex. Nash's celebrated result is the following:

Theorem 9.2. *(Nash Equilibrium)* Every game of pure strategy with continuous payoff functions and with closed, bounded and convex strategy sets has a Nash equilibrium. □

The proof of Theorem 9.2 is quite advanced. We observe here that, by our preceding discussion, Nash's Theorem implies the MiniMax Theorem in all cases. For the strategy sets of probability choices like S above are closed, bounded and convex as mentioned. Moreover, the payoff function is just the expected value which is easily seen to be continuous. Thus the Expected Value version of the game has a Nash equilibrium which is, by definition, the needed solution. We conclude with an example where the hypotheses of Nash's Theorem fail:

Example 9.3. Consider the "Sealed Bid Dollar Auction". Players 1 and 2 bid any dollar amount s and t with $s, t \geq 0$. The higher bidder receives the dollar after paying their bid while the lower bidder pays their bid and receives nothing. We declare ties go to Player 1. (Alternately, we could say both players pay and receive nothing.) It is easy to see that there is no Nash equilibrium for this game. (See Exercise 9.4.) Note that the strategy sets for both players are unbounded and thus the hypotheses of Theorem 9.2 are violated.

Exercises for Section 9

Exercise 9.1. Consider the following 3×3 payoff matrix corresponding to a zero sum game.

Player			P_2	
	Strategy	X	Y	Z
	A	-4	3	2
P_1	B	1	7	-5
	C	0	8	4

Eliminate dominated strategies to reduce this to a 2×2 payoff matrix and solve the corresponding game using the techniques of the previous section.

Exercise 9.2. Determine whether the game introduced in Example 9.1 has a Nash equilibrium. Justify your answer carefully.

Exercise 9.3. We consider another variation on the Baseball Card Auction. In this case, P_1 can bid any dollar amount s with $0 \leq s \leq 100$ and P_2 can bid any dollar amount with t with $10 \leq t \leq 90$. We assume that if both players bid the same amount then Player 1 wins the bidding. However, this time the losing bidder must pay their bid and receive nothing. Assuming P_1 values the card at \$75 dollars and P_2 at \$60, write down the payoff functions $p_1(s,t)$ and $p_2(s,t)$ as in Example 9.1. Decide whether this game has a Nash equilibrium. Justify your answer carefully.

Exercise 9.4. Explain carefully why the "Dollar Auction" as described in Example 9.3 does not have a Nash equilibrium. Justify your answer carefully.