PROBABILITY AND GAMES OF CHANCE

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1. Introduction

The roots of probability theory can be traced directly back to the study of parlor games and gambling. Take a pair of dice and roll them 12 times in succession. Should you bet on seeing double sixes at least once? How many rolls are required so that the odds of seeing double sixes (at least) once is better than $50-50$? This type of question arose in gaming circles in France in the 1650s. In a correspondence between perhaps the two most famous thinkers of the day, Blaise Pascal and Pierre Fermat, the question was resolved. In retrospect, this was the first theorem in the modern theory of probability.

Today, probability theory is prominent in every imaginable arena: from marketing to politics to medicine. Probabilities have become part of the vernacular as we discuss the stock-market, sports and even the weather. In these notes, we give an introduction to the elementary aspects of probability by focusing on the origins of the theory: the study of games of chance.

2. Games with Cards and Dice

We consider a couple of very simple examples of a games of chance which can be devised using a pair of fair dice or a deck of fair cards.

To begin, consider the following game: I roll a pair of fair dice. If I roll doubles (i.e., the two dice are equal) you win the game. Otherwise, I win the game. I propose that we make a “friendly” \$5 wager on this simple game. Should you play? Assuming that you are not completely opposed to gambling (or, for that matter, compulsively determined to gamble at any cost!), your decision here should be based on whether or not the game is fair, or better yet, in your favor. In other words, you would like to know the *probability* that I roll doubles. By convention, the probability of an event (such as “I roll doubles”) is a number between 0 and 1 whose size measures the likelihood of the event occuring. Here probability 1 means the event is certain to happen, probability 0 means the event is impossible while, say, probability $1/2$ means the even will happen, on average, 50% of the time. What is the probability that I roll doubles? To answer this question, note that it is important to treat the dice as separate entities, so we assume one is red and the other blue. We then see that there are 36 possible rolls 6 of which are doubles. We conclude that the probability of rolling doubles is $6/36 = 1/6$. Since you are only likely to win this bet, on average, one out of six times, you should probably refuse to take my friendly wager.

We can make this game a little bit more interesting. Again, I will roll the pair of dice. If I get doubles, you win while if I get a sum of nine then I win. Otherwise, I will roll again and keep rolling until one of these two outcomes occurs. What is the probability you win at this game? To answer this question, consider Figure 1:

Figure 1 : Rolling Dice

		Blue Dice					
		1	2	3	4	5	6
	1	XX					
	2		XX				
	3			XX			00
Red Dice	4				XX	00	
	5				00	XX	
	6			00			XX

XX = doubles

00 = sum 9

There are 6 ways to roll a doubles and 4 ways to roll a sum of nine. Since all the other possible rolls are irrelevant, we might as well restrict attention to these 10 rolls. We thus see that the probability of doubles is 6/10. This game favors you and you might care to wager on it!

Next, we consider a simple game with a standard deck of 52 cards. I shuffle and you draw 2 cards from the top of the deck without replacement. You win \$5 if you draw a pair of Jacks. Otherwise you pay me \$5 for the pleasure of playing. What are your chances in this game?

This game brings up an important issue. How should we think of a 2-card hand? On the one hand, we may wish to keep track of the order we received the cards. Thus a 2-card hand is an ordered pair (1st Card, 2nd Card) of 2 cards. With this view, we will distinguish, for example the outcome

$$\text{outcome } 1 = (\text{1st Card} = A\diamondsuit, \text{2nd Card} = J\spadesuit)$$

from

$$\text{outcome } 2 = (\text{1st Card} = J\spadesuit, \text{2nd Card} = A\diamondsuit).$$

How many possible 2-card hands are there? We have 52 possibilities for the 1st Card, and having chosen this card, 51 possibilities remaining for the 2nd Card giving $52 \cdot 51 = 2652$ possible ordered 2-card hands.

How many of these 2652 outcomes will win us \$5? To get a pair of Jacks, first we must choose our 1st Card to be a Jack: there are 4 choices. Then we must choose our 2nd Card to be another Jack: there now are 3 choices left. We conclude that $4 \cdot 3 = 12$ of the 2652 possible ordered 2-card hands will make us a winner. Our chances of winning the game are not good: Since $12/2652 = 1/221$, we can expect to win this game only once out of every 221 times we play.

There is an important alternative way to think of 2-card hands and to do the preceding calculation. In many card games (including this one) it is not the order in which you receive your cards that matters but simply the cards themselves. It is natural then to view the possible outcomes of this game as *sets* { 1st Card, 2nd Card } of 2 cards. We can then rely on the customary meaning of a set of objects and, particularly, what it means to say two sets are equal. Recall that two sets are equal if they have the same elements, regardless of the order or manner in which the elements are described. For example, the set of integers larger than 2 and the set of positive integers whose square is bigger than 4 are the same set. In our case, we can write

$$\text{outcome} = \{J\spadesuit, A\diamondsuit\}$$

for both of the ordered 2-card hands above.

How many 2-card hands are there? We could count the hands directly. Or we can simply observe that there are 1/2 as many outcomes as in the ordered case since each unordered set of 2 cards gives rise to exaclty 2 ordered hands. Thus there are $1/2 \cdot 52 \cdot 51 = 1326$ different outcomes.

How many of the 2-card hands consist of two Jacks? We can list them:

$$\{J\spadesuit, J\heartsuit\}, \{J\spadesuit, J\diamondsuit\}, \{J\spadesuit, J\clubsuit\}, \{J\heartsuit, J\diamondsuit\}, \{J\heartsuit, J\clubsuit\}, \{J\diamondsuit, J\clubsuit\}.$$

We could also have simply observed that there are 1/2 as many 2-element subsets as ordered pairs. In any case, in this setting we have 6 ways to win out of 1326 equally likely outcomes putting our chances, again, at 1/221.

These simple examples illustrate several features of probability theory that we will explore in these notes. The card game shows that computing probabilities is related to counting. In Section 4, we will explore this aspect of the theory in more detail. The second variation of the dice game illustrates a first calculation of what is known as *conditional probability.* We define this concept in Section 3 and use it to analyze the dice game "Craps" in Section 5.

Finally, we remark that our probability calculations gives rise to a precise measure of how much a game favors one player over another. For example, in the first game above (rolling doubles) a wager might be considered fair if your pay-off were 6 times mine, e.g., if you pay \$5 for losing but gain \$30 for winning. Alternately, we can take the \$5 you get for winning times the chance of winning this sum (1/6) and subtract the \$5 you stand to lose times the chance of losing (5/6). We arrive at a number

$$Ex = \$5 \cdot (1/6) - \$5(5/6) = \$5(1/6 - 5/6) = -\$3.33$$

called the *expected value* of the game for you. You can expect to lose \$3.33 per game, on average. Compare this with the second dice game in which $Ex = \$5(6/10 - 4/10) = +\$1.00$ (you should average a dollar a game profit) and the third game in which $Ex = \$5(1/221 - 220/221) = -\$4.95$ (you will lose your shirt at this one). Probabilities provide a (theoretical) cash register for the outcome of wagering.

Exercises for Section 2

1. Consider the following game which costs \$8 to play. Throw three dice. I pay you \$5 for each dice that shows a 6. What is the expected value of this game?

2. Consider the following game. You flip a coin. If you get heads, I give you \$2. The game is over. If not, you flip again. If you get heads on the 2nd flip, you get \$4 and, again, the game is over. If not flip a third time. If you get heads, you win \$8 and we're through or else you keep flipping. In general, I will pay you $\$2^n$ if get heads for the first time on the nth flip. Would you pay \$100 to play this game? Explain your answer in a couple of sentences. You will want to consider the expected value of this game, as defined above.

3. Basic Notions and Laws of Probability

In this section, we state precise definitions of the basic notions of probability theory and then give some first results concerning these definitions.

By a *probability experiment* we will mean any action or process whose execution results in exactly one of a number of well-determined and equally likely possible outcomes. We would like to think of rolling a dice, flipping a coin and dealing a hand of cards as examples of probability experiments. Note that we must be careful about describing the outcomes so that they are all equally likely. For example, when rolling a pair of dice we should agree that there are 36 possible outcomes as in Figures 1 and 2 above.

The *sample space* S of a probability experiment is the set of all possible outcomes. An *event* E in a probability experiment is a particular subset of the sample space. That is E is a particular, distinguished set of outcomes. We will often say an outcome in E is *favorable* to E. Finally, we define the *probability* $P(E)$ of E to be the fraction of the total number of outcomes in S that are in E. Precisely, any set A let

$$n(A) = \text{ the number of elements in } A.$$

Then we define

$$P(E) = \frac{n(E)}{n(S)}.$$

Example 3.1. Consider, as in the introduction, the probability experiment of rolling a pair of dice. The sample space S for this experiment is then the 36 possible rolls indicated in Figures 1 and 2. Let E be the event $E=$ "the roll is doubles". Then $n(E) = 6$ and so $P(E) = 6/36$. Similarly, if $F =$ "the sum is nine" then $n(F) = 4$ and $P(F) = 4/36$.

Example 3.2. Consider the experiment of drawing 2 cards from a standard deck of 52 cards. In this case, we have two possible ways to view the sample space as explained in the introduction. We may choose to think of the sample space as ordered pairs of 2 cards. In this case, the sample space contains $52 \cdot 51 = 2652$ different outcomes. Alternately, we may decide to treat each set of 2 cards as an outcome. In this case, our sample space is the collection of all 2-element subsets of the 52 cards. A formula for the number of (in general) k-element subsets from an n-element set will turn out to be a critical tool in our work in Section 4. In this case, we have observed that the sample space has $1/2 \cdot 52 \cdot 51 = 1326$ different outcomes.

Suppose now that we have a probability experiment with sample space S. Given two events E_1 and E_2 we can construct new events using the conjunctions "and" and "or" and the negation "not". Specificaly,

$$E_1 \,\text{or}\, E_2 = \text{all outcomes favorable to either } E_1,\, E_2 \text{ or both}$$

$$E_1 \,\text{and}\,\; E_2 = \text{all outcomes favorable to both } E_1 \text{ and } E_2 \text{ simultaneously}$$

$$\text{not}\, E_1 = \text{all outcomes unfavorable to } E_1$$

For example, in our experiment of rolling a pair of dice, we might let $E_1 =$ "roll is doubles" and $E_2 =$ "sum is 8". To compute $P(E_1 \,\text{or}\, E_2)$, we should count all the doubles (6) and all the eights (5) and add $6 + 5 = 11$. However, double 4's was counted twice and so we should subtract 1. Thus $n(E_1 \,\text{or}\, E_2) = 11 - 1$ and $P(E_1 \,\text{or}\, E_2) = 10/36$. Notice here that $E_1 \,\text{and}\;\, E_2 =$ "roll is double 4's" is precisely the event we subtracted off. it is even easier, to compute $P(\text{not}\, E_1)$. Since there are 6 doubles there must be $36 - 6 = 30$ non-doubles. Thus $P(\text{not}\, E_1) = 30/36 = 1 - P(E_1)$. This examples generalizes to

Theorem 3.3. *Let E_1 and E_2 be events in a probability experiment. Then*

(1) $P(E_1 \,or\, E_2) = P(E_1) + P(E_1) - P(E_1 \;and\;\, E_2)$

(2) $P(not\, E_1) = 1 - P(E_1)$

Proof. The proofs are counting arguments which can be visualized with Venn diagrams. Let S be the sample space. For (1) we have

$$\begin{aligned} P(E_1 \text{ or } E_2) &= \tfrac{n(E_1 \text{ or } E_2)}{n(S)} = \tfrac{n(E_1)+n(E_2)-n(E_1 \text{ and } E_2)}{n(S)} \\ &= P(E_1) + P(E_2) - P(E_1 \text{ and } E_2). \end{aligned}$$

For (2) we have

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{1 - n(\text{not } E_1)}{n(S)} = 1 - P(E_1)$$

□

We say two events E_1 and E_2 are *mutually exclusive* if $P(E_1 \text{ and } E_2) = 0$. Thus mutually exclusive events are those that cannot happen simultaneously, like $E_1 =$ "roll is doubles" and $F =$ "roll is nine" from Figure 2. Theorem 3.3 (1) yields the following

Corollary 3.4. (The Law of Mutually Exclusive Events) *Let E_1 and E_2 be mutually exclusive events. Then*

$$P(E_1 \, or \, E_2) = P(E_1) + P(E_1)$$

Corollary 3.4 answers the question: "When do we add probabilities?" Answering the related question "When do we multiply probabilities?" leads to a critically important concept in probability theory: namely, *conditional probability.*

To introduce the notion of conditional probability, consider the following simple game. I roll a pair of dice so that you cannot see. I tell you what the sum of the dice is, e.g. "sum = 6". You win \$5 if the roll is doubles otherwise you lose. The question we ask is: "What affect does the information I give you have on your willingness to play?" Let's examine this question in some special cases. Suppose I tell you the "sum is 2". Then you are guaranteed to have doubles! We will say the probability "roll is doubles" *given* the "sum is two" is 1. Notationally, this is written

$$P(\text{ "roll is doubles" } \mid \text{ sum is 2 }) = 1.$$

Here the symbol "|" reads as the word "given".

Let's compute in some other special cases: Suppose I tell you the "sum is 6". Then we may as well restrict our attention to the 5 ways to make a six. That is, writing B and R for the Blue and Red dice, our sample space can be effectively reduced to

$$\{B = 1, R = 5\}, \{B = 2, R = 4\}, \{B = 3, R = 3\}, \{B = 4, R = 2\}, \{B = 5, R = 1\}.$$

Now of these six rolls, only one is favorable to "roll is doubles". Thus

$$P(\text{ "roll is doubles" } \mid \text{ sum is 6 }) = 1/5.$$

The chances of winning are not so great but notice that they are slightly higher than they would be without the information "sum is 6". Contrast this with the case "sum is 7". There are now 6 rolls with sum 7 but obviously none are doubles. Thus

$$P(\text{ "roll is doubles" } \mid \text{ sum is 7 }) = 0.$$

Such examples suggest how to compute $P(E_1|E_2)$. First we restrict our attention the outcomes favorable to E_2. In other words, E_2 becomes the sample space since we know that one of these outcomes certainly occurred. We then count the events in E_2 favorable to E_1 also – that is we find $n(E_1 \text{ and } E_2)$. We then divide:

$$P(E_1|E_2) = \frac{n(E_1 \text{ and } E_2)}{n(E_2)}. \tag{1}$$

From this formula we can prove:

Theorem 3.5. (Law of Conditional Probability) *Let E_1 and E_2 be events in a probability experiment. Then*

$$P(E_1 \textit{ and } E_2) = P(E_1|E_2) \cdot P(E_2).$$

Proof. To prove this result, we "unsimplify" the fraction in expression (1) to create a complex fraction:

$$\begin{aligned} P(E_1|E_2) &= \frac{n(E_1 \text{ and } E_2)}{n(E_2)} \\ &= \frac{n(E_1 \text{ and } E_2)}{n(E_2)} \cdot \frac{\frac{1}{n(s)}}{\frac{1}{n(s)}} = \frac{\frac{n(E_1 \text{ and } E_2)}{n(S)}}{\frac{n(E_2)}{n(S)}} = \frac{P(E_1 \text{ and } E_2)}{P(E_2)} \end{aligned}$$

We then clear the denominator to get the desired formula. □

We can use Theorem 3.5 to give a satisfactory answer to the question "When do we multiply probabilities?" To do so we must introduce the notion of independent events. Informally, two events are independent events if the occurance of one has no affect on the occurance of the other. For example, if I flip a coin and then roll a pair of dice the events "coin is heads" and "roll is doubles" are independent.

For another, consider the experiment of drawing a card from a deck of 52. Let $E_1 =$ "Jack" and $E_2 =$ "Red card". then $P(E_1) = 1/13$. If E_2 occurs i.e. the card is red, then the chance of a Jack is still $2/26 = 1/13$. That is, $P(E_1|E_2) = P(E_1)$. We make this our formal definition:

Let E_1 and E_2 be two events in some probability experiment. We say E_1 and E_2 are *independent events* if $P(E_1|E_2) = P(E_1)$.

We can now answer our question "When do we multiply probabilites?" with the following theorem:

Theorem 3.6. (The Law of Independent Events) *Let E_1 and E_2 be independent events. Then*

$$P(E_1 \textit{ and } E_2) = P(E_1) \cdot P(E_2)$$

Proof. We just use Theorem 3.5 to write

$$P(E_1 \text{ and } E_2) = P(E_1|E_2) \cdot P(E_2)$$

and then substitute $P(E_1) = P(E_1|E_2)$ since E_1 and E_2 are independent events. □

We have now developed the basic laws of probability. We give some examples of how these simple ideas can be put to work.

Example 3.7. I have 10 fair dice. I will roll them simultaneously and give you \$5 if at least one of the dice comes up a 6. Otherwise, you will owe me \$5. Should you play? To decide, you use probabilties (of course). Note that the outcome of the roll of each of the 10 dice is independent of the other 9 dice. More precisely, let $E_1=$ "dice 1 is not a 6", $E_2 =$ "dice 2 is not 6", ..., $E_{10} =$ "dice 10 is not 6". Then

$$P(E_1) = P(E_2) = \cdots = P(E_{10}) = 5/6.$$

Moreover, $E_1, E_2, \ldots, E_{10}$ are mutually independent events. Thus

$$\begin{aligned} P(\text{“no dice is a 6”}) &= P(E_1 \text{ and } E_2 \text{ and } \cdots \text{ and } E_{10}) \\ &= P(E_1) \cdot P(E_2) \cdots P(E_{10}) \\ &= (5/6)^{10} \\ &= .1615 \end{aligned}$$

Thus your probability of wining \$5 is $P(\text{“at least one six”}) = 1 - .1615 = .8385$.

Example 3.8. What is the probability of rolling 6 dice simultaneously and getting each of the possible numbers 1 through 6 as your roll? We might call such a roll a "straight". To answer this question, it is useful to imagine the dice are all different collors, say Red, Blue, Green, Orange, Purple and White. Now one way of obtaining a straigh is to have $R = 1, B = 2, G = 3, O = 4, P = 5$ and $W = 6$. Let's call this straight "RBGOPW". What is the probability of getting this particular straight? We have

$$\begin{aligned} P(\text{“RBGOPW”}) &= P(R = 1 \text{ and } B = 2 \text{ and } \cdots \text{ and } W = 6) \\ &= P(R = 1) \cdot P(B = 2) \cdots P(W = 6) \\ &= (1/6)^6 \\ &= .00002 \end{aligned}$$

But "RBGOPW" is only one way to get a straight. We could also have "RBGOWP" and "BRGOPW" and many others. However, each of these straights have the same probability, namely .00002. Moreover, these events are all mutually exclusive. Thus $P(\text{“straight”}) = N \cdot (.00002)$ where N is the number of "straights". In Section 4, we find a formula for the number of ways to order k letters or objects – the number of *permutations* of a set. The answer may be familiar to you. Ask: How many ways to order the 6 letters R, B, G, O, P, W? Answer: First choose the first letter (6 choices), then the second letter (5 choices), then the third letter (4 choices) etc. until we have all 6 in order. Thus here are $N = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ orderings. This is often denoted $N = 6!$ (6 factorial). We conclude $P(\text{“straight”}) = 6! \cdot (.00002) = .0144$.

Exercises for Section 3

1. Compute the following probabilities for the experiment of rolling a pair of dice:

a. $P($ "doubles" and "sum at least 8"$) =$

b. $P($ "sum is odd" or "at least one dice is a 5"$) =$

c. $P($ "sum is at least 8" $|$ "at least one dice is a 5"$) =$

d. $P($ "sum is 6" $|$ "doubles"$) =$

e. Give an example of two independent events E_1 and E_2 for this experiment. Use the formal definition of independent events to justify your answer.

2. Compute the following probabilites for the experiment of drawing two cards (without replacement) from a deck of 52:

a. $P($ " a pair"$) =$

b. $P($ "both cards red"$) =$

c. $P($ "a pair" | "both cards red"$) =$

d. $P($ "both cards red" | "a pair" $) =$

3. Determine the probability of throwing a "straight" with 7 dice i.e. of throwing 7 dice and having the numbers 1 through 6 (plus one more number) appear.

4. Use the ideas developed so far to solve "Pascal's problem" mentioned above: Roll the dice once. What is the probability you *don't* get double sixes? Roll the dice twice in a row. What is the probability you don't get double sixes on either roll? Now generalize: Determine the number of rolls so that the probability of seeing double sixes on at least one roll is greater than .5.

4. Counting Cards in Bridge and Poker

Our ability to compute the probabilities which arise in simple games is often directly related to our ability to *count* occurances of various outcomes in rather complicated situations. The casino scene in the film Rain Man illustrates the possibilities for "counting cards" and winning at, in this case, Blackjack. A fast and precise photographic memory, as the character Raymond has in the movie, can improve the odds of winning over-all at Blackjack under casino rules to better than even. To win money, it is not sufficient just to be able to count the cards. Knowledge of the probabilities which arise as a shoe of cards is played out is critical to know when and how to bet. Here there is also "counting" going on but it is of a mathematical nature.

Consider the Poker game of 5-Card Draw. We will simplify matters by assuming that you are playing alone and that you are dealt 5 cards from a standard deck of 52. (A boring game yes, but not much worse than playing a slot machine!) A winning hand for you is a pair of Jacks or better but you need not know what this means yet. We are faced with a counting question regarding the possible 5-card hands. Again the issue of "to order" or not "to order" arises as in Example 3.2 above.

Let's begin by ordering, i.e., by keeping track of the order in which we receive our five cards C_1, C_2, C_3, C_4 and C_5. A particular hand then might be

$$C_1 = A\spadesuit, C_2 = 10\diamondsuit, C_3 = 2\diamondsuit, C_4 = Q\clubsuit, C_5 = Q\heartsuit.$$

This is a different hand from, say,

$$C_1 = Q\heartsuit, C_2 = 10\diamondsuit, C_3 = 2\diamondsuit, C_4 = Q\clubsuit, C_5 = A\spadesuit.$$

How many ordered 5-card hands are there? We can build one by first choosing C_1 (52 choices), then C_2 (51 choices), then C_3 (50 choices), then C_4 (49 choices) and then C_5 (48 choices). Every such sequence of choices yields a distinct hand. Thus we see that there are $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$ total ordered 5-card hands. Let us write $P(52, 5) = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$ and, more generally,

$$P(n, k) = n \cdot (n-1) \cdots (n-k+2) \cdot (n-k+1)$$

for $n \geq k \geq 1$. Then $P(n, k)$ counts the number of ways to build an ordered list of k objects from a set of n objects. The special case $n = k$ recovers the factorial function: $P(n, n) = n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$. This is simply the number of ways to order a set of n objects or the number of *permutations* of n objects.

Suppose we choose instead to view a 5-card hand without regard to order. We can then write both hands above (and many more like them) as the *set*

$$\{A\spadesuit, 10\diamondsuit, 2\diamondsuit, Q\clubsuit, Q\heartsuit\}.$$

We wish to count the number of 5-element subsets which can be chosen from the set of 52 cards. The formula in this case is not much harder but the answer is quite important. Observe that $P(52, 5)$ over-counts the number of unordered 5-card hands since, for example, the two ordered hands above are now regarded as the same. But here's a key question: How many ordered hands does $\{A\spadesuit, 10\diamondsuit, 2\diamondsuit, Q\clubsuit, Q\heartsuit\}$ represent? Certainly any ordering of these 5-cards will give a distinct ordered hand. Moreover, any ordered hand with these cards comes from one such ordering. Thus there are exactly $P(5, 5) = 5!$ ordered hands for every single unordered hand. We conclude that

$$\text{number of unordered 5-card hands} \quad = \frac{P(52, 5)}{P(5, 5)} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2,598,960.$$

In general, we define

$$\binom{n}{k} = \frac{P(n, k)}{P(k, k)} = \frac{n \cdot (n-1) \cdots (n-k+2) \cdot (n-k+1)}{k \cdot (k-1) \cdots 2 \cdot 1}.$$

The numbers $\binom{n}{k}$ are usually called the *binomial coefficients* because of their role as the coefficients in the expansion of binomials of the form $(x + a)^n$. They may also be familiar from Pascal's triangle. In fact, it was the gambling question discussed in the introduction (Exercise 2.4) which led to their discovery by Pascal. We will refer to the number $\binom{n}{k}$ as n *choose* k to highlight the role of these numbers in counting. We have seen that $\binom{n}{k}$ counts the number of k-element subsets that can be chosen from a set of n elements.

Example 4.1. Ten students meet in the fieldhouse to have a game of pick-up basketball. How many ways can two teams be chosen? Since there are 10 players and 5 per team, each time we choose one team the other is determined as well. So how many ways are there to choose a 5 player team from 10 players? The answer is

$$\binom{10}{5} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 252 \quad \text{ways.}$$

The game of Bridge is played with four players and a standard deck of 52 cards. The cards are dealt out completely so that each player has 13. We won't describe the play of the game, which would take us very far afield. We simply mention that the number of cards in each suit that you have in your hand becomes very important in Bridge. For example, if all 13 of your cards are the same suit (a very improbable event!), you can make a "Grand Slam". Suppose we draw 13 cards from a fair deck and find we have $S = 5$ spades, $H = 4$ hearts, $D = 2$ diamonds and $C = 2$ clubs. Note that the suit counts necessarily add up to 13. These numbers describe the *distribution* of our hand.

Example 4.2. We can ask a variety of probability questions concerning distributions of bridge cards. Let's consider the following one here: What is the probability of having exactly 8 cards in one suit in a 13 card bridge hand? Our formula for probabilities gives:

$$P(\text{“8 cards in some suit”}) = \frac{\text{number of hands with 8 cards in a suit}}{\text{number of 13 cards hands}}.$$

We know the denominator: There are $\binom{52}{13}$ 13-card hands. Notice that we are using *unordered* hands here. This will usually be our preference. We can compute the numerator, as follows: First we'll count the hands with exactly 8 spades, i.e. with S = 8. How do we build such a hand? First we pick the 8 spades from the 13 available – there are $\binom{13}{8}$ ways to pick do this. Next we need 5 more cards and none of them should be spades. So let's throw all the spades out of the deck leaving $52 - 13 = 39$ cards. We'll then choose our remaining 5 cards from these 39 – there are $\binom{39}{5}$ ways to do this. Thus there are $\binom{13}{8} \cdot \binom{39}{5}$ hands with exactly 8 spades. Finally, we observe that there are the same number of hands with exactly 8 hearts, with 8 diamonds and with 8 clubs. Thus we should multiply by 4 to get the number of hands with 8 in one suit. In conclusion,

$$P(\text{“8 cards in some suit”}) = \frac{4 \cdot \binom{13}{8} \cdot \binom{39}{5}}{\binom{52}{13}} = .00467.$$

Let's return to the game of Poker. As you may know, there are names for certain special hands: In ascending order of strength, we have *One Pair*, *Two Pair*, *Three of a Kind*, *Straight* (5 cards in ascending order, Aces can be low or high) *Flush* (all 5 cards the same suit), *Full House* (3 of a kind plus a pair) *Four of a Kind* and *Straight Flush* (5 cards in succession, all of the same suit). Of course, some 5-card hands could go by different names. For example, a Four of a Kind hand could be thought of as Two Pair. We will remove this ambiguity by always using the strongest name for a hand possible where strength is measured by the order above.

This brings us to an interesting question: Why are the 5-card Poker Hands ordered in this way? You may suspect the answer has to do with probabilities. As we will see, our list of Poker hands is in decreasing order of probability: That is, we have:

Theorem 4.3.

$$P(\text{One Pair}) > P(\text{Two Pair}) > \cdots > P(\text{Four of a Kind}) > P(\text{Straight Flush})$$

Proof. The proof is obtained by computing each of the probabilities in question directly. We will take this up in the next example and in the exercises. $\square$

Example 4.4. We compute the probabilities of some of the special hands and leave the remaining for the exercises.

Straight Flush: This one is the easiest of all. A Straight Flush is determined by a suit (4 choices) and then the lowest card which can be an Ace through a 10, since 10 J Q K A is the highest straight, (10 choices). Thus there are 40 total straight flushes and

$$P(\text{Straight Flush}) = \frac{40}{\binom{52}{5}} = .000015.$$

Flush: How do we describe a Flush? We first specify what suit all of our cards are (4 choices) and then say what 5-cards of this suit we have. Now once we fix a suit, we are restricted to 13 card values. Thus there are $\binom{13}{5}$ possible flushes in each fixed suit and $4 \cdot \binom{13}{5}$ total flushes. Now we must be careful, since we don't want to count the straight flushes again. However, we know there are 36 of these and so we can subtract to get $4 \cdot \binom{13}{5} - 40$ flushes. We conclude:

$$P(\text{ Flush}) = \frac{4 \cdot \binom{13}{5} - 40}{\binom{52}{5}} = .00197.$$

Three of a Kind: As in the case of a flush, it is useful to imagine describing a typical hand (in this case, a Three of a Kind) to try to count all such hands. So how would we describe a hand with Three of a Kind? First we might say which card value (2 through Ace) we have three of: there are 13 possibilities here. Next we might say which suits our three are in: there are 4 suits and we need to choose 3 so there are $\binom{4}{3} = 4$ possibilities here. Finally, we will indicate our last two cards. Here we must be a little careful. We don't want either to be the 4th remaining card of our three of a kind. Thus we should reduce our choices to the 48 cards which remain, when we exclude these four. Now we also don't want to get a pair when we choose 2 cards from these 48, for then we would have a Full House. So we ask how many ways are there to choose two cards but not a pair from 48? Well there are $\binom{48}{2}$ possible 2-card hands. How many are pairs? To get a pair, we first pick one of the remaining 12 values and then choose the two suits (4 suits, choose 2 or $\binom{4}{2} = 6$ choices). Thus there are $\binom{48}{2} - 12 \cdot 6 = 1056$ ways to choose the last two cards. We conclude that

$$P(\text{Three of a Kind}) = \frac{4 \cdot 13 \cdot 1056}{\binom{52}{5}} = .02113.$$

Exercises for Section 4

1. Compute the probability of drawing 5 cards and getting Four of a Kind.

2. Compute the probability of drawing 5 cards and getting a Full House.

3. Compute the probability of drawing 5 cards and getting a Straight

4. Compute the probability of drawing 5 cards and getting Two Pair.

5. Compute the probability of drawing 5 cards and getting One Pair.

6. Compute the probability of having two 6 card suits in a hand of Bridge.

5. Probability Trees and the Game of Craps

Many games have branching features wherein the outcome of one stage of the game affects the odds in the subsequent stages. A famous example of this is the game Craps in which the opening roll or point determines the criteria for winning. (We will examine this game carefully below.) The idea of a branching game can be applied to tournaments as well. For example, in the NCAA basketball field of 64 we may think of each round as a stage in a large game. A team's chances of winning at each stage is determined not just by their strength but by the opponents they meet at each stage.

To understand the probabilities in a game with this type of branching it is necessary to consider conditional probabilities as discussed in Section 4. Fortunately, we have an excellent device at our disposal for keeping things organized – a *probability tree.*

We illustrate the use of a probability tree, by considering a simple game. You flip a coin twice. Let H be the number of heads you get so that $H = 0, 1$ or 2. I give you one dice if $H = 0$, two dice if $H = 1$ and three dice if $H = 2$ and you roll your dice. You win \$5 if the sum of the dice you throw equals 5 and lose \$5 with any other sum.

To compute the probability of winning this game, it helps to consider the various possibilities for the coin toss individually. Suppose we throw no heads so that $H = 0$. Then we only have one dice to roll and our chances of getting a 4 are $1/6$. In the language of conditional probability we can say

$$P(\text{“sum} = 5\text{”} \,|\, \text{“}H = 0\text{”}) = 1/6.$$

If $H = 1$ then we throw two dice and our chances of getting a sum of 5 is $4/36 = 1/12$. Thus

$$P(\text{“sum} = 5\text{”} \,|\, \text{“}H = 1\text{”}) = 1/12.$$

What if $H = 2$? Then we roll three dice. There are now 3 ways to get a sum of 5 $= 2 + 2 + 1$ (i.e. in which 2 of the dice are a 2 and the other a 1) and also 3 ways to get a sum of $5 = 3 + 1 + 1$. Since there are 6^3 possible rolls for three dice we conclude

$$P(\text{“sum} = 5\text{”} \,|\, \text{“}H = 2\text{”}) = 6/6^3 = 1/36.$$

So we know the probability of winning the game in each of the possible scenarios, but how do convert this into a probability of winning outright? Notice that, logically, a winning outcome must belong to one of the three events: “sum = 5” and "$H = 0$", "sum $= 5$" and "$H = 1$, or "sum $= 5$" and "$H = 2$. Morever, these events are mutually exclusive. Thus

$$\begin{aligned} P(\text{“sum is 5”}) = P(&\text{“sum} = 5\text{” and “}H = 0\text{”}) \\ &+ P(\text{“sum} = 5\text{” and “}H = 1\text{”}) \\ &+ P(\text{“sum} = 5\text{” and “}H = 2\text{”}). \end{aligned}$$

Now the event "sum is 5" and the event "$H = 0''$ (or $H = 1, 2$) are certainly not independent events. By design, the probability of "sum is 5" changes depending on the value of H. However, we have the formula:

$$P(E_1 \text{ and } E_2) = P(E_1|E_2) \cdot P(E_2)$$

(Theorem 3.5). Here this becomes

$$\begin{aligned} P(\text{“sum} = 5\text{” and “}H = 0\text{”}) &= P(\text{“sum} = 5\text{”}|\text{“}H = 0\text{”}) \cdot P(\text{“}H = 0\text{”}) \\ P(\text{“sum} = 5\text{” and “}H = 1\text{”}) &= P(\text{“sum} = 5\text{”}|\text{“}H = 1\text{”}) \cdot P(\text{“}H = 1\text{”}) \\ P(\text{“sum} = 5\text{” and “}H = 2\text{”}) &= P(\text{“sum} = 5\text{”}|\text{“}H = 2\text{”}) \cdot P(\text{“}H = 2\text{”}). \end{aligned}$$

Thus

$$\begin{aligned} P(\text{“sum is 5”}) = P(&\text{“sum} = 5\text{”}|\text{“}H = 0\text{”}) \cdot P(\text{“}H = 0\text{”}) \\ &+ P(\text{“sum} = 5\text{”}|\text{“}H = 1\text{”}) \cdot P(\text{“}H = 1\text{”}) \\ &+ P(\text{“sum} = 5\text{”}|\text{“}H = 2\text{”}) \cdot P(\text{“}H = 2\text{”}). \end{aligned}$$

It is straightforward to compute $P(\text{“}H = 0\text{”}) = 1/4, P(\text{“}H = 1\text{”}) = 1/2$ and $P(\text{“}H = 2\text{”}) = 1/4$ and so we conclude:

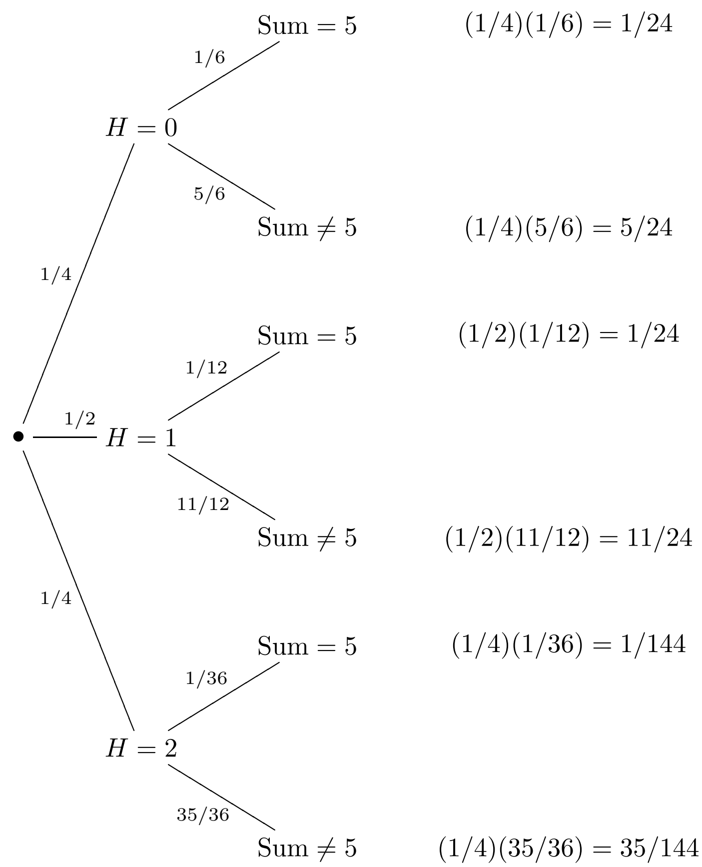
$$\begin{aligned} P(\text{“sum is 5”}) &= (1/4) \cdot (1/6) + (1/2) \cdot (1/12) + (1/4) \cdot (1/36) \\ &= .0903 \end{aligned}$$

We can visualize this calculation with a probability tree (Figure 3, below). The tree opens from left to right with the root node $\bullet$ representing the beginning of the game. The first event is the flip of the coin. We follow *branches* to the three possible outcomes or *nodes* $H = 0, 1$ or 2 which can occur here. We label the branches with the probabilities of reaching the nodes.

The next stage is the roll of the dice. In this case, we are only interested in two possible outcomes of this roll: Sum $= 5$ or Sum $\neq 5$. We label the branches to these two types of nodes with the probabilities of reaching them. Notice it is here that we are implicitly looking at conditional probabilities. For example, the probability of reaching the node Sum $= 5$ from the node $H = 0$ is precisely the conditional probability $P(\text{“sum} = 5\text{”}|\text{“}H = 0\text{”}) = 1/6$.

The probability of traversing any path from left to right in the tree is obtained by multiplying the branches. Thus, for example, the probability that the game will result in you throwing 2 Heads and then rolling a sum other than five with three dice is 35/144. The tree facilitates correct and easy use of the Law of Conditional Probability. It is easy to imagine using a tree for branching games with three or more stages Notice, however, that the number of nodes of the tree grows exponentially with the number of stages and so writing down a complete tree quickly becomes unfeasible.

Figure 3: A Probability Tree



We can put the probability tree to good use in analyzing a the game "Craps". Played on a long green velvet table with one roller and many side betters and spectators, "Craps" is the classic casino dice game. The game can go on indefinitely, in principle, with the tension and side bets mounting with each throw. The rules are simple: The roller throws the dice. If she rolls a sum of 7 or 11 she wins. If she throws a sum of 2, 3 or 12 ("craps") she loses. Otherwise call the sum she rolls the *point* P. The roller continually rolls until either (i) she rolls the sum P again or (ii) she rolls a 7. In the first case the roller wins the bet and in the second she loses. Spectators may bet with the roller or on a number of side bets listed on the table.

Our goal, or more correctly your goal, will be to compute the probability of winning the game of Craps. To tackle this problem, we can make one simplifying observation: Although a game of Craps entails many rolls (potentially), we can actually treat the game as a two stage branching game. What are the stages?

The first stage is, not surprisingly, the first roll of the dice. The possible outcomes of this roll will represent the first nodes in our tree. The events we should consider are Roll = 7 or11, Roll = 2,3 or 12, $P = 4$, $P = 5$, $P = 6$, $P = 8$, $P = 9$, $P = 10$. In the first event, we win, in the second we lose. In the other 6 we have rolled our point.

We now enter the second stage of the game. Here's the key idea for our analysis. While it may take a long while, we may be sure that on some roll you will either roll your point again and win or roll 7 or craps and lose the game. For simplicity, lets call the first event Roll Point and the second Roll 7. At each of the "point" nodes ($P = 4$ through $P = 10$) must compute a conditional probability. We can then complete the probability tree for Craps and determine the probability of winning the game.

Exercises for Section 5

1. Compute the the conditional probabilities

$$P(\text{Roll Point}|P = n)$$

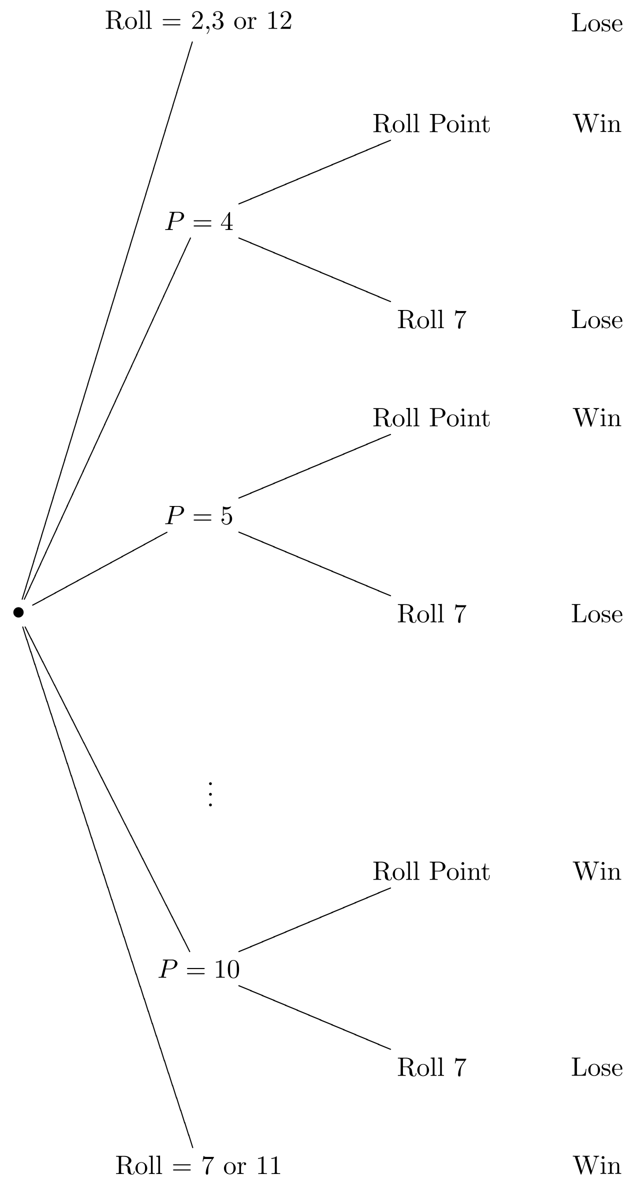
for $n = 4, 5, 6, 8, 9$ and 10.

2. Write down the complete probability tree for the game Craps and compute the probability of winning the game. See Figure 4, below for a partial rendering of the final tree.

3. In a casino, the "Don't Win" bet wins if, on the first roll, a 3 or 12 is thrown and loses if, on the the first roll, a 7 is thrown. If a 2 is thrown on the first roll the bet is a "push" – no one wins. If neither a $2, 3, 7, 11$ nor 12 is rolled on the first roll then the first roll is called the "point". The "Don't Win" bet wins if the shooter throws a 7 before the point and loses if the shooter throws the point before a 7. Compute the probability of winning a "Don't Win" bet.

4. You play a game of craps and win. What is the probability that you rolled a 6 on your first roll. Hint: Use Theorem 3.5 to compute $P(\text{Point} = 6|\text{Win})$.

Figure 4: Probability Tree for the Game Craps


Roll = 2,3 or 12
Lose
Roll Point
Win
$P = 4$
Roll 7
Lose
Roll Point
Win
$P = 5$
Roll 7
Lose
Roll Point
Win
$P = 10$
Roll 7
Lose
Roll = 7 or 11
Win