Rational Types of Function Space Components for Flag Manifolds

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July 20, 1995

1. Introduction.

2. Preliminaries. In this section, we establish notation for use with Sullivan’s differential graded algebra (DGA) approach to rational homotopy theory. Our references for rational homotopy theory are [3, 19].

By a DGA \((A, d)\) we mean a connected commutative graded algebra \(A\) over \(\mathbb{Q}\) together with a degree one derivation \(d\) whose square is zero. Given a rational vector space \(V^i\) we denote by \(\Lambda_i(V^i)\) the free, graded algebra generated in degree \(i\) by the elements of \(V^i\). Alternately, if \(x_1, \ldots, x_n\) is a basis for \(V^i\) we write \(\Lambda_i(V^i) = \Lambda_i(x_1, \ldots, x_n)\). If \(A\) can be written, as algebra, in the form \(A = \bigotimes_{i=0}^\infty \Lambda_i(V^i)\) then \((A, d)\) is free and we may write \(A = \Lambda(V),\) where \(V = \bigoplus_{i=0}^\infty V^i\) is the corresponding graded vector space. Conversely, if \(A = \Lambda(V)\) for some graded space \(V\) we let \(\text{Vect}_i(A) = V^i,\) the vector space of degree \(i\) generators of \(A\). Recall a DGA \((A, d)\) is minimal if \(A\) is a free graded algebra whose generating vector space admits a well-ordered, homogeneous basis \(x_\alpha\) such that \(dx_\alpha\) is a polynomial without linear term in the \(x_\beta\) with \(\beta < \alpha.\)

Given a nilpotent complex \(X,\) a DGA \((A, d)\) is a model for \(X\) if there is a chain map from \((A, d)\) to the de Rham-Sullivan DGA of rational forms on \(X\) which induces an isomorphism on cohomology. A model \((A, d)\) is the minimal model for \(X\) if \((A, d)\) is a minimal DGA. Sullivan proved the existence and uniqueness (up to DGA isomorphism) of a minimal model for \(X\) as hypothesized. He showed, moreover, that the minimal model of \(X\) is a

Key words and phrases. Sullivan Minimal Model, Flag Manifold
unique invariant of the rational homotopy type of $X$. We denote the minimal model of $X$ by $(\mathcal{M}_X, d_X)$.

We will restrict our attention, in the sequel, to classes of spaces $X$ whose minimal models are of a particularly accessible form. At the very least we will assume $X$ is an $F_0$-space; that is, $X$ is a simply-connected complex with finite-dimensional (rational) homotopy and homology satisfying $H^\text{odd}(X) = 0$. The minimal model $(\mathcal{M}_X, d_X)$ for such an $X$ is a two-stage DGA. Specifically, $(\mathcal{M}_X, d_X)$ can be written in the form

$$\mathcal{M}_X = \Lambda(V_0) \otimes_{d_X} \Lambda(V_1)$$

where $V_0$ is evenly graded, $V_1$ oddly graded and where the differential $d_X$ satisfies $d_X|V_0 = 0$ and $d_X(V_1) \subseteq \Lambda(V_0)$. In fact $X$ is a hyperformal space [8] which means there exists a homogeneous basis $y_1, \ldots, y_n$ for $V_1$ so that $d_X(y_1), \ldots, d_X(y_n)$ is a regular sequence in the graded algebra $\Lambda(V_0)$. One attractive feature of the class of $F_0$-spaces is the applicability of the following

\textbf{Theorem 2.1} [1, Corollary 3.6] If $X$ and $Y$ are $F_0$-spaces then the set of rational homotopy classes of maps $[X, Y]$ is in bijection with the group of algebra maps $\text{Hom}(H^*(Y), H^*(X))$. \(\square\)

Examples of $F_0$-spaces include products of even-dimensional spheres, complex projective spaces and, more generally, homogeneous spaces of the form $G/H$ for $G$ a connected Lie group and $H$ a closed subgroup of maximal rank [9]. We will especially consider the subclass of $F_0$-spaces consisting of spaces $X$ whose cohomology is generated in degree two. Note that the minimal model for such an $X$ is two-stage of the form

$$\mathcal{M}_X = \Lambda_2(x_1, \ldots, x_m) \otimes_{d_X} \Lambda(V_1)$$

where $V_1$ has a homogeneous basis $y_1, \ldots, y_m$ such that $d_X(y_i) = P_i(x_1, \ldots, x_m)$ a homogeneous polynomial in the $x_j$ and where $P_1, \ldots, P_m$ is a regular sequence in the polynomial algebra $\Lambda_2(x_1, \ldots, x_m)$. Examples of spaces in this subclass are products of complex projective spaces and flag manifolds of the form $G/T$ where $G$ is a connected Lie group and $T$ is a maximal torus.

As a final preliminary we record the following useful lemma whose proof is straight-forward:
Lemma 2.2 Let \((\mathcal{A}, d)\) be a two-stage DGA of the form

\[ \mathcal{A} = \Lambda_2(x_1, \ldots, x_m) \otimes_d (\Lambda_1(w_1, \ldots, w_k) \otimes \Lambda(V_1)), \]

where \(V_1^{\leq 3} = 0\). Suppose \(d(w_1) = x_1, \ldots, d(w_r) = x_r\) while \(d(w_i) = 0\) for \(i > r\). Suppose \(d(y_i) = P_i(x_1, \ldots, x_m)\) where \(y_1, \ldots, y_n\) is some homogeneous basis for \(V_1\). Let \((\mathcal{M}, \delta)\) be the minimal DGA given by

\[ \mathcal{M} = (\Lambda_1(\bar{w}_{r+1}, \ldots, \bar{w}_k) \otimes \Lambda_2(\bar{x}_{r+1}, \ldots, \bar{x}_m)) \otimes_\delta \Lambda(V_1), \]

where \(\delta(\bar{w}_{r+i}) = 0\), \(\delta(\bar{x}_{r+i}) = 0\) and \(\delta(y_i) = P_i(0, \ldots, 0, \bar{x}_{r+1}, \ldots, \bar{x}_m)\). Then there exists a DGA homomorphism from \((\mathcal{M}, \delta)\) to \((\mathcal{A}, d)\) inducing an isomorphism on cohomology. In particular, if \((\mathcal{A}, d)\) is a model for a space \(X\) then \((\mathcal{M}, \delta)\) is the minimal model for \(X\). \(\square\)

3. Sullivan Models for Function Spaces. In this section, we use the Hirsch Lemma [3] to deduce the existence of a two-stage model for components of the space of maps between two \(F_0\)-spaces. We then apply results of Haefliger to describe the differential explicitly in the case the terminal space is cohomologically generated in degree two. Our arguments follow the line of the proof of [17, Theorem 3]

Theorem 3.1 Let \(f : X \to Y\) be a map between \(F_0\)-spaces. Write the minimal model for \(Y\) as \(\mathcal{M}_Y = \Lambda(V_0) \otimes_{d_Y} \Lambda(V_1)\), where we recall \(V_0\) is evenly graded and \(V_1\) is oddly graded. Then there exists a model \(\mathcal{A}_f\) for the function space \(M_f(X, Y)\) of the form

\[ \mathcal{A}_f = \Lambda(Z_0) \otimes_{d_f} \Lambda(Z_1), \]

where

\[ Z_0^n = \bigoplus_{i=0}^{\infty} H_i(X) \otimes V_0^{i+n} \quad \text{and} \quad Z_1^n = \bigoplus_{i=0}^{\infty} H_i(X) \otimes V_1^{i+n}. \]

Proof. Recall [10, Theorem B] that if \(e : Y \to Y_\mathbb{Q}\) is the rationalization map for \(Y\) then the induced map \(e : M_f(X, Y) \to M_{e\circ f}(X, Y_\mathbb{Q})\) is the rationalization for \(M_f(X, Y)\). We may thus assume that \(Y\) is a rational space.

By the correspondence between Postnikov towers and minimal models (see [3]), \(Y\) can be viewed as the total space of a principal fibration with fibre and base products of rational Eilenberg-Maclane spaces. Specifically,
let $K_0 = \Pi_i K(\text{Hom}(V^i_0, \mathcal{Q}), i)$ and $K_1 = \Pi_i K(\text{Hom}(V^i_1, \mathcal{Q}), i)$. Then there exists a pull-back diagram of fibrations

\[
\begin{array}{ccc}
Y & \longrightarrow & K_1 \\
\downarrow p & & \downarrow p_\infty \\
K_0 & \longrightarrow & BK_1 \\
\end{array}
\]

where $p_\infty$ is the path/loop fibration over $BK_1 = \Pi_i K(\text{Hom}(V^i_1, \mathcal{Q}), i+1)$. Observe that the space $M(X, K_1)$ is connected since $[X, K_1] = \bigoplus_i H^i(X) \otimes V^i_1$ and $H^*(X)$ is evenly graded while $V_i$ is oddly graded. Thus from (1) we obtain a fibration diagram of function spaces of the form

\[
\begin{array}{ccc}
M_f(X, Y) & \longrightarrow & M(X, K_1) \\
\downarrow p & & \downarrow p_\infty \\
M_{p\circ f}(X, K_0) & \longrightarrow & M_0(X, BK_1) \\
\end{array}
\]

We claim that (2) is a pull-back diagram as well. Using the fact that (1) is a pull-back diagram, we can identify the total space $E$ of the pull-back of $p_\infty$ by $k$ as $E = \{g : X \rightarrow Y | p \circ g \simeq p \circ f\}$. Thus we must show that $E = M_f(X, Y)$. To see this, first observe that $M_f(X, Y)$ is a full component of $E$. Conversely, note that the obstructions to lifting a homotopy between $p \circ g \simeq p \circ f : X \rightarrow K_0$ to a homotopy between $g$ and $f$ lie in the groups $H^n(X, \pi_n(K_1))$. Again, since $H^*(X)$ is evenly graded while $V_i$ is oddly graded these groups are trivial and the claim follows.

We next recall the classical result of Thom in [20] which asserts that for any CW complex $X$, any abelian group $\pi$ and any map $f : X \rightarrow K(\pi, n)$ there is an equivalence $M_f(X, K(\pi, n)) \simeq \Pi_{i \geq 0} K(H^{n-i}(X, \pi), i)$. Using this fact and the natural identification $M(X, PK_1) \approx PM(X, K_1)$ of $M(X, PK_1)$ as the path space on $M(X, K_1)$ we see that $p_\infty$ is a principal fibration. Since $p$ is a pull-back of $p_\infty$ it follows that the cohomology of $M(X, K_1)$ is transgressive.
with respect to \( p \) (see [3, p. 257]). We may thus apply the Hirsch Lemma ([3, Lemma 4.1]) to the fibration \( p \) to obtain a two-stage model \((\mathcal{A}_f, d_f)\) for \( M_f(X, Y) \) of the form

\[
\mathcal{A}_f = H^*(M_{p_0f}(X, K_0)) \otimes_{d_f} H^*(M_0(X, K_1)).
\]

The theorem now follows from Thom’s result. \( \square \)

Now suppose that \( Y \) is cohomologically generated in degree two so that \((\mathcal{M}_Y, d_Y) = \Lambda_2(x_1, \ldots, x_m) \otimes_{d_Y} \Lambda(V_1)\). Given \( f : X \to Y \) where \( X \) is any \( F_0 \)-space note that, in this case, the model \((\mathcal{A}_f, d_f)\) may be written in the form \( \mathcal{A}_f = \Lambda_2(x_1, \ldots, x_m) \otimes_{d_f} \Lambda(Z_1) \). where \( Z_1^0 = \oplus_{i=0}^{\infty} H_{2i}(X) \otimes V_1^{2i+n} \).

In this special case, we can identify the differential \( d_f \) explicitly. Given an element \( b \in H_{2i}(X) \) we view \( b \) as an element of the dual space \( H^{2i}(X) \) and write \( b(a) \in \Phi \) for the value of \( b \) on \( a \in H^{2i}(X) \). Let \( a_i = f^*(x_i) \in H^2(Y) \) for \( i = 1, \ldots, m \), where we here identify \( x_i \) with its image in \( H^2(Y) \). We will write \( c = (c_1, \ldots, c_m) \) to denote an \( m \)-tuple of nonnegative integers and put \( |c| = \sum_{i=1}^{m} c_i \). With this notation we prove

**Theorem 3.2** Let \( f : X \to Y \) be a map between \( F_0 \)-spaces with \( Y \) cohomologically generated in degree two. Let \((\mathcal{A}_f, d_f)\) be the above model for \( M_f(X, Y) \). Then given \( b \otimes y \in H_{2i}(X) \otimes V_1^{2i+n} \subset Z_1^0 \) write \( d_Y(y) = P(x_1, \ldots, x_m) \) for some homogeneous polynomial \( P(x_1, \ldots, x_m) \in \Lambda_2(x_1, \ldots, x_m) \). Then \( d_f(b \otimes y) \in \Lambda_2(x_1, \ldots, x_m) \) is given by the formula

\[
d_f(b \otimes y) = \sum_{|c|=i} (c_1! \cdots c_m!) \cdot b(a_1^{c_1} \cdots a_m^{c_m}) \cdot \frac{\partial^{|c|}}{\partial x_1^{c_1} \cdots \partial x_m^{c_m}} P(x_1, \ldots, x_m).
\]

**Proof.** From the proof of Theorem 2.1 and the Hirsch Lemma we have

\[
d_f(b \otimes y) = k^*(b \otimes y) \in H^*(M_{p_0f}(X, K_0)) \cong \Lambda_2(x_1, \ldots, x_m),
\]

where we recall that \( k : M_{p_0f}(X, Y) \to Y \) is the map induced by the generalized \( k \)-invariant of \( Y \) (see (1) and (2) above). Note that, identifying \( V_1 = H^*(K_1) \), we have \( k^*(y) = d_Y(y) \in H^*(K_0) \).

In [6], Haefliger showed how to describe induced maps such as \( k \) on cohomology. Following [17, §7], let \( \varepsilon_{p_0f} : X \times M_{p_0f}(X, K_0) \to K_0 \) be the evaluation map. Then by [17, Lemma 7.2] given \( x_i \in H^2(K_0) \) we have

\[
\varepsilon_{p_0f}^*(x_i) = 1 \otimes x_i + (p \circ f)^*(x_i) \otimes 1 = 1 \otimes x_i + a_i \otimes 1.
\]
(Recall we are identifying the elements $x_i \in H^2(K_0)$ with their images $p^*(x_i) \in H^2(Y)$ under the algebra inclusion.) Now given an element $b \in H_*(X)$ and an element $a \otimes P \in H^*(X \times M_{pof}(X, K_0))$ write

$$b \cap (a \otimes P) = b(a)P \in H^*(K_0).$$

With this notation and Haefliger's result (see [17, Lemma 7.1]) we have

$$d_f(b \otimes y) = k^*(b \otimes y)$$
$$= b \cap \varepsilon_{pof}^*(k^*(y))$$
$$= b \cap \varepsilon_{pof}^*(P(x_1, \ldots, x_m))$$
$$= b \cap P(\varepsilon_{pof}^*(x_1), \ldots, \varepsilon_{pof}^*(x_m))$$
$$= b \cap P(a_1 \otimes 1 + 1 \otimes x_1, \ldots, a_m \otimes 1 + 1 \otimes x_m).$$

The proof now follows from the identity

$$P(a_1 \otimes 1 + 1 \otimes x_1, \ldots, a_m \otimes 1 + 1 \otimes x_m)$$

$$= \sum_{|c| \leq |P|} (c_1! \ldots c_m!) a_1^{i_1} \ldots a_m^{i_m} \otimes \frac{\partial |c|}{\partial x_1^{c_1} \ldots \partial x_m^{c_m}} P(x_1, \ldots, x_m).$$

\(\Box\)

We conclude this section by describing explicitly the rational homotopy types of two distinguished components of function spaces: the space of null maps $M_0(X,Y)$ from $X$ to $Y$ and the space $M_1(X,X)$ of self-maps of $X$ which are homotopic to the identity. The rational homotopy type of $M_0(X,Y)$ can be determined explicitly for appropriate $X$ and $Y$ using Theorem 3.2. Specifically,

**Theorem 3.3** Let $X$ and $Y$ be $F_0$-spaces with $Y$ cohomologically generated in degree two. Let $k$ be the degree of the largest nontrivial homotopy group of $Y$. For each odd $n = 1, 3, 5, \ldots, k-2$ set $a_n = \sum_{i>0} \dim H_i(X) \cdot \dim \pi_{i+n}(Y)$. Then

$$M_0(X,Y) \simeq \mathbb{Q} Y \times (S^1)^{a_1} \times (S^3)^{a_3} \times \cdots \times (S^{k-2})^{a_{k-2}}.$$

**Proof.** By Theorem 2.2, the differential $d_0$ in the model $(A_0, d_0)$ for $M_0(X,Y)$ is given on an element $b \otimes y \in Z_1^n = \bigoplus_{i=0}^\infty H_i(X) \otimes V_1^{i+n}$ by

$$d_0(b \otimes y) = 0 \text{ for } b \in H_{>0}(X) \text{ and } d_0(1 \otimes y) = d_Y(y).$$
Thus \((A_0, d_0) \cong (M_Y, d_Y) \otimes \Lambda(\tilde{Z}_i)\) where \(\tilde{Z}_i^n = \bigoplus_{i=0}^{\infty} \tilde{H}_i(X) \otimes V_i^{i+n}\). \(\Box\)

**Remark.** Theorem 3.3 expresses the fact that, for \(X\) and \(Y\) as hypothesized, 
\[M_0(X, Y) \cong Y \times M_0(X, Y)\] where \(M_0(X, Y)\) is the space of basepoint-preserving null maps from \(X\) to \(Y\). In other words, the evaluation fibration is rationally decomposable for such \(X\) and \(Y\). See [17, Theorem 3] which gives this result for null self-maps of rational two-stage Postnikov systems, and [7] for examples of the failure of this result integrally.

The space \(M_1(X, X)\) is a topological monoid and so rationally a product of Eilenberg-MacLane spaces. Determining the rational homotopy groups of \(M_1(X, X)\) for \(X\) an \(F_0\)-space was shown by Meier [14, Theorem A] to be equivalent to proving the Halperin conjecture true for such \(X\). In [16], Shiga and Tezuka confirmed the Halperin conjecture for homogeneous spaces \(G/H\) where \(G\) is a compact connected Lie group and \(H\) a closed subgroup of maximal rank. Combining, these results we obtain

**Theorem 3.4** (Meier, Shiga and Tezuka) Let \(X = G/H\) where \(G\) is a compact connected Lie group and \(H\) is a closed subgroup with rank \(H = \text{rank } G\). Let \(k\) be the degree of the largest nontrivial homotopy group of \(X\). For each odd \(n = 1, 3, 5, \ldots, k\) set 
\[b_n = \sum_{i \geq 0} (\dim H_i(X) - \dim H_{i-1}(X)) \cdot \dim \pi_{i+n}(Y).\]

Then \(M_1(X, X) \cong \bigotimes (S^1)^{b_1} \times (S^3)^{b_3} \times \cdots \times (S^k)^{b_k}\).

**Proof.** Taken together, [16, Theorem A] and [14, Theorem A] imply the rational homotopy of \(M_1(X, X)\) is oddly graded. Thus, by Theorem 2.1, for odd \(n = 1, 3, \ldots, k\) we must have \(\dim \pi_n(M_1(X, X)) = \dim Z_1^n - \dim Z_0^{n+1}\). The result now follows. \(\Box\)

**4. Some Classification Theorems.** In this section, we use the model of Section 3 to deduce some classification theorems for the rational types represented by function space components. To begin we consider the space of self-maps of a flag manifold \(X\); that is, a homogeneous space of the form \(G/T\) where \(G\) is a compact, connected Lie group and \(T\) is a maximal torus. The cohomology of \(X\) is, in this case, given by the classical result of Borel [2]. Specifically, let \(B = \Lambda_2(x_1, \ldots, x_m)\) be the graded polynomial algebra in \(m = \text{rank } G\) variables and let \(W\) denote the Weyl group of \(G\). Let \(W\) act on \(B\) by permuting the subscripts of the \(x_i\). Then \(H^*(X) = B/J\) where \(J\) is the ideal of \(B\) consisting of polynomials invariant under \(W\).
Regarding the ideal $J$ recall that the Weyl group $W$ of $G$ is a finite
reflection group and so can be viewed as a subgroup of the orthogonal
group $O(n)$. It follows that the polynomial $P(x_1, \ldots, x_m) = x_1^2 + \cdots + x_m^2$ is invariant
under $W$ and so an element of $J$ of grade four. Notice that $P$ has the property
that its $m$ partial derivatives $\frac{\partial}{\partial x_1}P(x_1, \ldots, x_m), \ldots, \frac{\partial}{\partial x_m}P(x_1, \ldots, x_m)$ span
the vector space $H^2(X) = \mathcal{Q}(x_1, \ldots, x_m)$. We use this last observation to prove

**Theorem 4.1** Let $X_1 = G_1/T_1$ and $X_2 = G_2/T_2$ be flag manifolds with $G_1$
simple. Let $f : X_1 \to X_2$ be a given map. Then $f^* : H^* (X_2) \to H^* (X_1)$ is
either trivial or surjective.

**Proof.** With notation as above, write $H^* (X_i) = B_i/J_i$, $i = 1, 2$, where $B_1 = \Lambda_2(x_1, \ldots, x_m)$ and $B_2 = \Lambda_2(y_1, \ldots, y_n)$. Let $P_1(x_1, \ldots, x_m) = x_1^2 + \cdots + x_m^2 \in J_1$ and $P_2(y_1, \ldots, y_n) = y_1^2 + \cdots + y_n^2 \in J_2$. We would like to assert that, since
$G_1$ is simple, $P_1$ is (up to scalar multiple) the unique element of grade four in
$J_1$. If $G_1$ is simply connected this is true and can be read, for example, from
the known values of degrees of basic invariants (see [11, p. 59]). Alternately,
the uniqueness of $P_1$ in the simply connected case follows from Borel’s results
and the classical result of E. Cartan that dim $\pi_3 (G_1) = 1$. If $G_1$ is not simply
connected, the assertion as stated is not true since the element $x_1 + \cdots + x_m$ of
grade two appears in $J_1$. However, if we replace $B_1$ by $B'_1 = \Lambda_2(x_1, \ldots, x_{m-1})$
and $J_1$ by the appropriate ideal $J'_1$ and set $x_m = -x_1 - \cdots - x_{m-1}$, then the
polynomial $P_1(x_1, \ldots, x_m) = x_1^2 + \cdots + x_m^2$ appears and is the unique element
(up to multiples) of grade four in $J'_1$. For convenience we will continue to write
$H^* (X_1) = B_1/J_1$ in either case.

Now suppose $f^* : H^* (X_2) \to H^* (X_1)$ is nontrivial. Let $a_i = f^*(y_i)$, $i = 1, \ldots, n$. Let $\phi : B_2 \to B_1$ be the map determined by $f^*$. Then $\phi(\mathcal{J}_2) \subseteq \mathcal{J}_1$
and so, by the uniqueness of $P_1$ in $J_1$, we must have $\phi(P_2) = \alpha P_1$ for some
$\alpha \in \mathcal{Q}$. Observe that if $\alpha = 0$ we obtain

$$a_1^2 + \cdots + a_n^2 = \phi(P_2(y_1, \ldots, y_n)) = 0$$

which implies each $a_i = 0$, contrary to our assumption. Thus $\alpha \neq 0$ and we
have a nontrivial identity of the form

$$P_2(a_1, \ldots, a_n) = \alpha P_1(x_1, \ldots, x_m).$$

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Now each \( a_i \) may be viewed as a linear endomorphism \( a_i(x_1, \ldots, x_m) \) of the vector space \( \mathcal{Q}(x_1, \ldots, x_m) \). With this view, we can use the chain rule to take the partial derivative \( \frac{\partial}{\partial x_i} \) of both sides of (3) and obtain

\[
\alpha \cdot \frac{\partial}{\partial x_i} P_1(x_1, \ldots, x_m) = \sum_{j=1}^{n} \frac{\partial}{\partial y_j} P_2(a_1, \ldots, a_n) \cdot \frac{\partial}{\partial x_j} a_i(x_1, \ldots, x_m).
\]

As observed above, the partials \( \frac{\partial}{\partial x_i} P_1(x_1, \ldots, x_m) \) for \( i = 1, \ldots, m \) span the vector space \( H^2(X_1) = \mathcal{Q}(x_1, \ldots, x_m) \). On the other hand, clearly,

\[
\frac{\partial}{\partial y_j} P_2(a_1, \ldots, a_n) \cdot \frac{\partial}{\partial x_j} a_i(x_1, \ldots, x_m) \in \mathcal{Q}(a_1, \ldots, a_n)
\]

for all \( i, j \). We conclude that \( \mathcal{Q}(a_1, \ldots, a_n) = H^2(X_1) \) which means \( f^* : H^2(X_2) \to H^2(X_1) \) is surjective. Since \( X_1 \) is cohomologically generated in degree two, it follows that \( f^* : H^*(X_2) \to H^*(X_1) \) is itself surjective. \( \square \)

Theorem 4.1 taken together with Theorem 2.1 implies the problem of determining the set of rational homotopy classes of self-maps of certain flag-manifolds \( X \) amounts to determining the group \( E(X_{\mathcal{Q}}) \) of self-equivalences of the rationalization of \( X \). Specifically, we have

**Corollary 4.1** Let \( X = G/T \) be flag-manifold with \( G \) simple. Then

\[
[X_{\mathcal{Q}}, X_{\mathcal{Q}}] = E(X_{\mathcal{Q}}) \cup \{0\}. \quad \square
\]

We remark that \( E(X_{\mathcal{Q}}) \) is given explicitly by [12, Corollary 2] when \( G = U(n) \), the unitary group.

Now, for any space \( X \), if \( f : X \to X \) is a self-equivalence then a homotopy inverse for \( f \) induces an equivalence \( M_f(X, X) \simeq M_1(X, X) \). Thus, Corollary 4.2 implies

**Corollary 4.2** Let \( X = G/T \) be a flag manifold with \( G \) simple. Then the components of the function space \( M(X, X) \) represent exactly two distinct rational homotopy types.

**Box** Note that the rational homotopy types of these two components – \( M_0(X, X) \) and \( M_1(X, X) \) – are given explicitly by Theorems 3.3 and 3.4, respectively.
We next consider the rational classification problem for function space components for other homogeneous spaces cohomologically generated in degree two. Of course in order to obtain a complete theorem like Corollary 4.3, we must restrict our attention to spaces $X$ and $Y$ for which the set $[X, Y]$ is known. For example, if $Y$ is a product of complex projective spaces then, for any $F_0$-space $X$, maps $f : X \to Y$ are classified by the “heights” in $H^*(X)$ of the images of the generators of $H^*(Y)$. Given an element $a \in H^*(X)$ we define the height of $a$ to be zero if $a = 0$ and otherwise by $\text{ht}(a) = \max \{n|a^n \neq 0\}$. Thus the following result—which is complementary to [15, Theorem 3.2]—represents the complete classification of the components of $M(X, Y)$:

**Theorem 4.2** Let $Y = \mathbb{CP}^{n_1} \times \cdots \times \mathbb{CP}^{n_k}$ and let $X$ be any $F_0$-space. Given a map $f : X \to Y$ set $h_j = \text{ht}(f(x_j))$ for $j = 1, \ldots, k$ where $x_j \in H^2(\mathbb{CP}^{n_j})$ is nontrivial. Define integers $a_{2i+1, j}$ for $i = 0, \ldots, n_j$, $j = 1, \ldots, k$ by first setting $a_{2i+1, j} = \dim H_{2(n_j-i)}(X)$ and then letting $a_{2i+1, j} = a_{2i+1, j}'$, for $i \neq n_j - h_j$, and $a_{2i+1, j} = a_{2i+1, j}' - 1$, for $i = n_j - h_j$. Define spaces $A_j$ by

$$A_j = \mathbb{CP}^{n_j-h_j} \times (S^1)^{a_1} \times \cdots \times (S^{2n_j-2})^{a_{2n_j-1}}.$$  

Then

$$M_f(X, Y) \simeq \varphi A_1 \times \cdots \times A_k.$$  

**Proof.** Let $f_j : X \to \mathbb{CP}^{n_j}$ denote $f$ composed with the $j$th projection. We must show that $M_{f_j}(X, \mathbb{CP}^{n_j}) \simeq \varphi A_j$. The minimal model for $\mathbb{CP}^{n_j}$ is of the form $\Lambda_2(x_j) \otimes \Lambda(y_{n_j})$ with $|y_{n_j}| = 2n_j - 1$ and $d(y_{n_j}) = x_j^{n_j+1}$. By Theorem 3.1, $\mathcal{A}_{f_j} = \Lambda_2(x_j) \otimes \Lambda(Z_1)$, where $Z_1$ is the oddly graded space with $Z_i^{2+1} = H_{2(n_j-i)}(X) \otimes \mathbb{Q}(y_{n_j})$ for $i = 0, \ldots, n_j$. Let $b_m \in H^2m(X)$ be dual to the element $f^*(x_j^m) \in H^2m(X)$. By Theorem 3.2, the image of $d_{f_j}$ in $\Lambda_2(x_j)$ is generated by the elements

$$d_{f_j}(b_m \otimes y_{n_j}) = \frac{d^m}{dx^m} x_j^{n_j+1} = (n_j + 1) \cdots (n_j - m + 1) \cdot x_j^{n_j-m+1},$$

for $m = 0, \ldots, h_j$. Thus the the image of $d_f$ is generated by the monomial $x_j^{n_j-h_j+1}$ and the conclusion follows. $\square$
In [5], the authors determine the set of rational self-maps for a large class of generalized complex flag manifolds. A special case of their results combined with our methods provide for an extension of Corollary 4.3. Specifically,

**Theorem 4.3** Let \( X = U(n+2)/T^2 \times U(n) \) for \( n \geq 1 \). Then the components of the space \( M(X, X) \) represent exactly two distinct rational homotopy types.

**Proof.** If \( n \) is odd the result is a direct consequence [5, Theorems 1.3,1.4] which imply \([X, X] = E(X, X) \cup \{0\}\). Assume \( n \) is even and let \( f : X \to X \) be any map. The minimal model \((M_X, d_X)\) for \( X \) is of the form \( M_X = \Lambda_2(x_1, x_2) \otimes_{d_X} \Lambda(y_{n+1}, y_{n+2}) \), where \( |y_m| = 2m - 1 \) and \( d_X(y_m) = T_m(x_1, x_2) \). Thus the model \((A_f, d_f)\) for \( M_f(X, X) \) is of the form \( A = \Lambda_2(x_1, x_2) \otimes \Lambda(Z_1) \). It suffices to show that if \( f \) rationally nontrivial then \( d_f : Z_1 \to \Lambda_2(x_1, x_2) \) is surjective. Of course, if \( f \) is a rational equivalence this is just Theorem 3.4. If \( f \) is not a rational equivalence then by [5, Theorem 1.4] \( f \) is a “projective map”. That is, swapping subscripts if necessary, \( f^* : H^2(X) \to H^2(X) \) satisfies \( f^*(x_1) = \alpha x_1 \) and \( f^*(x_2) = -\alpha x_1 \) for some \( \alpha \neq 0 \). Composing \( f \) with a self-equivalence of \( X \) if necessary, we may assume \( \alpha = 1 \). By [5, Proposition 4.1], \( \text{ht}(x_i) = n + 2 \). Thus we have nontrivial elements \( b_k \in H_{2k}(X) \) dual to \( x_1^k \) in \( H^{2k}(X) \) for \( k = n + 1, n + 2 \). Using Theorem 3.2, we compute

\[
d_f(b_n \otimes y_{n+1}) = \sum_{\left| c \right| = n} (c_1!c_2!b_n(x_1^{c_1}(-x_1)^{c_1}) \frac{\partial^n}{\partial x_1 \partial x_2} T_{n+1}(x_1, x_2)
\]

\[
= \sum_{\left| c \right| = n} (c_1!c_2!) (-1)^{c_2} [(c_1 + 1)!c_2!x_1 + c_1!(c_2 + 1)!x_2].
\]

Rewriting this last expression, we obtain

\[
d_f(b_n \otimes y_{n+1}) = C(n) \cdot (x_1 + x_2),
\]

where

\[
C(n) = \sum_{c_2=0}^{n} [(n - c_2)!c_2!]^2 [c_2 + 1](-1)^{c_2}.
\]

A similar calculation shows \( d_f(b_n \otimes y_{n+1}) = C(n + 1) \cdot (x_1 - x_2) \). Thus we must show that \( C(m) \neq 0 \) for all \( m > 0 \). In fact, comparing like terms it is easy to see that \( C(m) < 0 \) for \( m \) odd and \( C(m) > 0 \) for \( m \) even. \( \Box \)

**Remark.** In [5], the authors conjecture that self-maps of spaces \( U(n)/H \) with \( H \) is a closed subgroup of maximal rank can be rationally classified as either trivial, projective or an equivalence. Since, by [5, Theorem 3.1] there are no projective self-maps of the space \( U(n + q) / T^q \times U(n) \) for \( q > 2, n > 1 \) the extension of Theorem 4.4 to this latter class of spaces would follow trivially.

5. **Function Spaces for Flag Manifolds.** In this section, we determine the rational homotopy types of function space components for some
specific homogeneous spaces. To begin, we consider the class of complex flag manifolds \( X_n = U(n+1)/T^{n+1} \) for \( n \geq 1 \). Define a sequence of homogeneous polynomials \( T_m \) for \( m \geq 1 \) by the formula

\[
T_m(x_1, \ldots, x_n) = \sum_{|c| = m} x_1^{c_1} \cdots x_n^{c_n},
\]

where we again write \( c = (c_1, \ldots, c_n) \) for an \( n \)-tuple of nonnegative integers and put \( |c| = \sum_{i=1}^n c_i \). Borel’s results in [2] imply the minimal model \((\mathcal{M}_{X_n}, d_{X_n})\) for \( X_n \) is of the form

\[
\mathcal{M}_{X_n} = \Lambda_2(x_1, \ldots, x_n) \otimes d_{X_n} \Lambda(y_2, \ldots, y_{n+1})
\]

where \( |y_m| = 2m - 1 \) and \( d_{X_n}(y_m) = T_m(x_1, \ldots, x_n) \). (See also [5, §2].)

Regarding the polynomials \( T_m \), we prove

**Lemma 4.1** Let \((d_1, \ldots, d_k)\) be a \( k \)-tuple of nonnegative integers with \( k \leq n \) and suppose \( m \geq |d| \). Then

\[
\frac{\partial^{|d|}}{\partial x_1^{d_1} \cdots \partial x_k^{d_k}} T_m(x_1, \ldots, x_n) = \sum_{|c|=m-|d|} (c_1 + d_1)! \cdots (c_k + d_k)! \ x_1^{c_1} \cdots x_n^{c_n}.
\]

**Proof.** Write \( T_m \) in the form

\[
T_m(x_1, \ldots, x_n) = (x_1^{d_1} \cdots x_k^{d_k}) \cdot \sum_{|c|=m-|d|} x_1^{c_1} \cdots x_n^{c_n} + \sum x_1^{c_1} \cdots x_n^{c_n},
\]

where the second sum ranges over nonnegative \( n \)-tuples satisfying \(|c| = m\) and \( c_i \neq d_i \) for some \( i = 1, \ldots, k \). Observe that \( \frac{\partial^{|d|}}{\partial x_1^{d_1} \cdots \partial x_k^{d_k}} \) kills all terms in the second sum and the result follows easily. \( \square \)

By Theorem 4.1 and its corollaries, the components of the function space \( M(X_k, X_n) \) are all equivalent when \( k > n \) and fall into two distinct classes when \( k = n \). The situation for \( k < n \) is much more complicated – we know of no classification theorem for rational homotopy classes of maps between \( X_k \) and \( X_n \). While a complete classification of the components of \( M(X_k, X_n) \) is thus out of reach, we give instead a classification of components corresponding to what we call “simple” maps and thereby identify many nonequivalent rational types represented by this space. Write \( H^2(X_k) = \mathbb{Q}(t_1, \ldots, t_k) \) and \( H^2(X_n) = \mathbb{Q}(x_1, \ldots, x_n) \). Then we make the following
**Definition.** We say a map $f : X_k \to X_n$ is *simple* if there exists $\alpha \in \mathcal{Q}$ so that for each $i = 1, \ldots, n$, $f^*(x_i) \neq 0$ implies $f^*(x_i) = \alpha \cdot t_j$ for some $j$.

Simple maps from $X_k$ to $X_n$ have the advantage of admitting classification. For each $l = 0, 1, \ldots \lceil n/k \rceil$, we construct a simple map as follows: Define an algebra map $\phi : \Lambda_2(x_1, \ldots, x_n) \to H^*(X_k)$ by letting

$$
\phi(x_1) = \cdots = \phi(x_l) = t_1, \ldots, \Phi(x_{(k-1)l+1}) = \cdots = \Phi(x_{lk}) = t_k
$$

and $\Phi(x_i) = 0$ for $i > lk$. Since $\Phi$ clearly carries symmetric functions in the $x_i$ to symmetric functions in the $t_j$, it induces a map $f^* : H^*(X_n) \to H^*(X_k)$. Applying Theorem 2.1 we obtain a map $f : X_k \to X_n$ which is simple by construction. Let us call such maps $f$, and more generally maps of the form $g \circ f \circ h$ with $g$ and $h$ rational equivalences of $X_k$ and $X_n$, respectively, *simple of type $l$*. It is clear that the simple maps of type $l$ are rationally distinct. The following fact is also easy, and we omit the proof.

**Lemma 4.2** Let $f : X_k \to X_n$ be a simple map. Then there exists $l \in \{0, \ldots, \lceil n/k \rceil \}$ so that $f$ is simple of type $l$.

We now turn to the problem of classifying the components of $M(X_k, X_n)$ corresponding to simple maps. Thus fix $k$ and $n$ with $2 \leq k \leq n$ and $l \in \{1, \ldots, \lceil n/k \rceil \}$. We define a sequence of integers $a_{2m+1}$ for $m = 1, \ldots, n+1$.

First set

$$a'_{2m-1} = \sum_{j=0}^{(n-m+1)} \dim H_{2j}(X_k)$$

and let $a_1 = a'_1 - (k+2)$. Next let $m \geq 2$ and set $a_{2m-1} = a'_{2m-1} - (k+1)$ for $m < \min\{l, n-lk\}$, $a_{2m-1} = a'_{2m-1} - k$ for $n-lk < m \leq l$, $a_{2m-1} = a'_{2m-1} - 1$ for $l < m \leq n-lk$, and $a_{2m-1} = a'_{2m-1}$ for $m > \max\{l, n-lk\}$. We prove

**Theorem 4.4** Let $f : X_k \to X_n$ be simple of type $l > 0$. Then

$$M_f(X_k, X_n) \simeq_{\mathcal{Q}} X_{n-k-1} \times (X_{l-1})^k \times (S^1)^{a_1} \times \cdots \times (S^{2n+1})^{a_{2n+1}}.$$

*Proof.* By Lemma 5.1, we may assume $f$ is the map constructed above. Let $(\mathcal{A}_f, d_f)$ be the model of §3 for $M_f(X_k, X_n)$. Then $\mathcal{A}_f = \lambda(x_1, \ldots, x_n) \otimes_{d_f} \Lambda(Z_i)$ where $Z_i$ is the oddly graded space given by

$$Z_i^{2m-1} = \bigoplus_{i=0}^{n-m+1} H_{2i}(X) \otimes \mathcal{Q}(y_{i+m}).$$

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The idea of the proof is to split the list of variables \(\{x_1, \ldots, x_n\}\) into \(k + 1\) sublists, namely, \(\{x_1, \ldots, x_l\}, \ldots, \{x_{l(k-1)+1}, \ldots, x_{lk}\}\) and \(\{x_{lk+1}, \ldots, x_n\}\) and show that the image of \(d_f\) is precisely the ideal \(J\) of \(\Lambda_2(x_1, \ldots, x_n)\) generated by function symmetric in each of these variable lists separately.

To be more explicit, we adopt the convention that for \(i = 1, \ldots, k\) \(T_{m,i} = T_m(x_{(i-1)k+1}, \ldots, x_{ik})\) and similarly write \(T_{m,k+1} = T_m(x_{k+1}, \ldots, x_n)\). It is then unambiguous to assume for the remainder of the proof that \(T_m\) (with a single subscript) means \(T_m(x_1, \ldots, x_n)\).

Let 
\[
J = (T_{1,1}, \ldots, T_{l,1}, \ldots, T_{1,k}, \ldots, T_{l,k}, T_{1,k+1}, \ldots, T_{n-lk,k+1})
\]
be the indicated ideal of \(\Lambda_2(x_1, \ldots, x_n)\). Observe that \(J\) is the ideal generated by functions symmetric in each of these variable lists separately and that we have expressed \(J\) with a minimal generating set. Applying Lemma ?? (*** normal form ***) and Theorem ?? (*** \(H_2(X_{n-1})\) ***) our Theorem will follow from a dimension count, once we show \(d_f(Z_1) = J\).

To show \(J \subseteq d_f(Z_1)\) it is convenient to use a different generating set for \(J\).

Given \(m > 0\), let \(P_m\) be the degree \(m\) polynomial defined by \(P_m(x_1, \ldots, x_k) = \sum_{i=1}^k x_i^m\). As above, write \(P_{m,i}\) for \(P_m\) applied to our \(i\)th variable list, \(i = 1, \ldots, k\). An easy induction then shows
\[
J = (P_{1,1}, \ldots, P_{l,1}, \ldots, P_{1,k}, \ldots, P_{l,k}, T_1, \ldots, T_{2n+1}).
\]

Regarding the differential \(d_f\) in grade one, let \(b_i \in H_2(X_k)\) be dual to \(t_i \in H^2(X_k)\) so that \(b_i \otimes y_2 \in Z^1_1\). From the definition of \(f\) and Theorem 3.2, we have
\[
d_f(b_i \otimes y_2) = \sum_{j=l(i-1)+1}^i \frac{\partial}{\partial x_j} T_2 = P_{1,i} + T_1.
\]
Since \(k \geq 2\), \(t_i^2 \in H^4(X_k)\) is nontrivial. Let \(b_{1,1}\) be dual to this element so that \(b_{1,1} \otimes y_3 \in Z^1_1\). Observe that
\[
d_f(b_{1,1} \otimes y_3) = \sum_{j=1}^l \frac{\partial^2}{\partial x_j^2} T_3 = 4P_{1,1} + 2T_1.
\]
Thus \(d_f(Z^1_1)\) contains the grade two elements of \(J\).

Next observe that, for \(m = 2, \ldots n+1\), \(1 \otimes y_m \in Z^{2m-1}_1\) and \(d_f(1 \otimes y_m) = T_m\). To show the elements \(P_{m,i}\) are in the image of \(d_f\) we first prove that
\[
\frac{\partial}{\partial x_i} T_{m+1} \equiv x_i^m \mod(T_1, \ldots, T_m).
\]
The case $m = 0$ is trivial and the general result follows by induction and the identity
\[ \frac{\partial}{\partial x_i} T_{m+1} = 2 \frac{\partial}{\partial x_i} x_i \cdot T_m = T_m + x_i \cdot \frac{\partial}{\partial x_i} T_m. \]

By Theorem 3.2 again we thus have
\[ d_f(b_i \otimes y_m) = \sum_{j=(i-1)+1}^{i} \frac{\partial}{\partial x_j} T_{m+1} \equiv P_{m,i} \text{mod}(T_1, \ldots, T_m), \]
and so $J \subseteq d_f(Z_1)$.

To complete the proof, we must show the reverse inclusion. Let $c = (c_1, \ldots, c_k)$ be any $k$-tuple with $|c| \leq n$ and let $b_c \in H_{2|c|}(X_k)$ be dual to the monomial $t_1^{c_1} \cdots t_k^{c_k} \in H^{2|c|}(X_k)$. Given $m$ satisfying $|c| < m \leq n + 1$, we see that the elements $b_c \otimes y_m$ span $Z_1^{2(|c|-1)}$. Thus it suffices show that $d_f(b_c \otimes y_m)$ is symmetric in each of our $k + 1$ variable lists separately.

Let $d = (d_1, \ldots, d_l)$ denote an $l$-tuple of nonnegative integers and define a sequence of linear operators $D_i$ for $i = 1, \ldots, k$ by
\[ D_i = \sum_{|d| = c_i} (d_1! \cdots d_l!) \frac{\partial^{d_l}}{\partial x_{l(i-1)+1} \cdots \partial x_{li}^{d_l}}. \]

Now $D_i$ is clearly invariant under permutations in the variables $\{x_{l(i-1)+1}, \ldots, x_{li}^{d_l}\}$ and, of course, trivially in our other $k$ lists. It follows that if $T(x_1, \ldots, x_n)$ is any homogeneous polynomial which is symmetric in each of our variable lists separately then $D_i(T)$ is also such a polynomial. To complete the proof then, observe that by Theorem 3.2 we have $d_f(b_c \otimes y_m) = D_k \circ \cdots \circ D_1(T_m)$.

\[ \square \]

References


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