RATIONALIZED EVALUATION SUBGROUPS OF A MAP: SULLIVAN MODELS, DERIVATIONS, AND G-SEQUENCES

GREGORY LUPTON AND SAMUEL BRUCE SMITH

ABSTRACT. We identify the homomorphism induced on rational homotopy groups by the evaluation map \( \omega : \text{map}(X, Y; f) \to Y \), in terms of a map of complexes of derivations constructed using Sullivan minimal models. Our identification allows for the characterization of the rationalization of the \( n \)th evaluation subgroup of \( f \), that is, of the image of \( \omega_\# \) in \( \pi_n(Y) \). It also allows for the identification of several long exact sequences of rational homotopy groups, including the long exact sequence induced on rational homotopy groups by the evaluation fibration. As a consequence, we obtain an identification of the rationalization of the so-called \( G \)-sequence of the map \( f \). We use these results to study the \( G \)-sequence in the context of rational homotopy theory. We give new examples of non-exact \( G \)-sequences and uncover a relationship between the homology of the rational \( G \)-sequence and negative derivations of rational cohomology. We also relate the splitting of the rational \( G \)-sequence of a fibre inclusion to a well-known conjecture in rational homotopy theory.

1. Introduction

Suppose given a based map \( f : X \to Y \) of simply connected CW complexes. Denote by \( \text{map}(X, Y; f) \) the path component of the space of (unbased) maps \( X \to Y \) consisting of those maps that are homotopic to \( f \). Then evaluation at the basepoint of \( X \) gives a based map \( \omega : \text{map}(X, Y; f) \to Y \). We refer to this map as the evaluation map. We define the \( n \)th evaluation subgroup of \( f \) to be the subgroup \( G_n(Y, X; f) = \omega_\#(\pi_n(\text{map}(X, Y; f))) \) of \( \pi_n(Y) \). The famous Gottlieb groups \( G_\ast(X) \) occur as the special case in which \( X = Y \) and \( f = 1_X \) [6]. The Gottlieb groups of a space have been much studied by homotopy theorists (see [23] for a survey of results and references). While many general results are known, explicit computation of \( G_\ast(X) \) appears difficult and is limited to a small number of sporadic examples. One reason that accounts in part for this difficulty is the fact that a map of spaces \( f : X \to Y \) does not necessarily induce a corresponding homomorphism of Gottlieb groups, since in general \( f_\#(G_n(X)) \nsubseteq G_n(Y) \). In particular, attempts to study \( G_\ast(X) \) via a cell decomposition of \( X \) are frustrated, since it is not clear what effect a cell attachment may have on the Gottlieb groups. One tool for studying \( G_\ast(X) \) that attempts to circumvent this problem is the so-called \( G \)-sequence of a map, introduced by Woo and Lee in [38].

The \( G \)-sequence of a map \( f : X \to Y \) is a sequence of groups and homomorphisms that derives from the long exact homotopy sequence of the map \( f \). It arises as
follows. One has the following commutative diagram of spaces:

\[
\begin{array}{ccc}
\text{map}(X, X; 1) & \xrightarrow{f_*} & \text{map}(X, Y; f) \\
\downarrow \omega & & \downarrow \omega \\
X & \xrightarrow{f} & Y
\end{array}
\]

The induced homomorphisms of homotopy groups \((f_*)_{\#}\) and \(f_{\#}\) form part of the long exact homotopy sequences of the maps \(f_*\) and \(f\) respectively. The evaluation maps induce maps of each term in these long exact sequences, resulting in a homotopy ladder. The \(G\)-sequence of the map \(f\) is then the image of the top long exact homotopy sequence in that of the bottom, shown as the middle row here:

\[
\begin{array}{cccccccc}
\cdots & \xrightarrow{\pi_{n+1}(f_*)} & \xrightarrow{\omega_{\#}} & \pi_n(\text{map}(X, X; 1)) & \xrightarrow{(f_*)_{\#}} & \pi_n(\text{map}(X, Y; f)) & \xrightarrow{\omega_{\#}} & \cdots \\
\cdots & \xrightarrow{G_{n+1}^{rel}(Y, X; f)} & \xrightarrow{G_n^{rel}(X)} & \xrightarrow{f_{\#}} & G_n(Y, X; f) & \xrightarrow{\omega_{\#}} & \cdots \\
\cdots & \xrightarrow{\pi_{n+1}(f)} & \xrightarrow{f_*} & \pi_n(X) & \xrightarrow{f_{\#}} & \pi_n(Y) & \xrightarrow{\omega_{\#}} & \cdots
\end{array}
\]

Here, \(G_{n+1}^{rel}(Y, X; f)\) denotes the “relative” term, defined as the image in \(\pi_{n+1}(f)\) of the homomorphism that derives from the evaluation maps. The homomorphisms in the \(G\)-sequence are restrictions of those in the long exact homotopy sequence of the map \(f\). Therefore, the \(G\)-sequence forms a chain complex (consecutive compositions are trivial). Some general conditions are known under which the \(G\)-sequence is exact (e.g. [14, 26]), but in general it is not exact (e.g. [14]).

In this paper we bring the techniques of rational homotopy theory to bear on problems and questions concerning evaluation subgroups (of a map) in general, and the \(G\)-sequence in particular. Our main goal is to expand the range of application of these techniques in this area. To this end, we are primarily concerned with establishing a suitable framework for considering such questions. At the same time, we obtain a number of results of interest in their own right.

The paper is organized as follows: Our main results are established in Sections 2 and 3. The basic result in Section 2 is Theorem 2.1, in which we identify the map induced on rational homotopy groups by the evaluation map \(\omega: \text{map}(X, Y; f) \to Y\). We describe this induced homomorphism as the map induced on homology by a map of complexes of derivations of the Sullivan minimal models of \(X\) and \(Y\). Theorem 2.1 has a number of immediate corollaries. For instance, we obtain a characterization of the rationalized evaluation subgroups of a map (Corollary 2.6) that extends a well-known characterization, due to Félix and Halperin, of the (rationalized) Gottlieb groups of a space in terms of derivations of its minimal model.

In Section 3, we extend and amplify the basic result of Theorem 2.1. We show that several related long exact sequences of rational homotopy groups may be expressed as long exact sequences in homology of derivation complexes of Sullivan minimal models (Theorems 3.3, 3.5, and 3.12). In Corollary 3.13 we obtain a
description of the rational homotopy groups of the based mapping space. In Theorem 3.8, we obtain a description, within the framework of derivation spaces, of the $G$-sequence of a map after rationalization.

In Section 4, we use the framework established in Sections 2 and 3 to study questions concerning the rationalized $G$-sequence. The antecedents of these questions begin with Gottlieb who observed that a map of spaces does not necessarily induce a map of Gottlieb groups [6, Sec.1]. While Gottlieb defined the evaluation subgroups of a map in [6], he did not study them as such. A number of basic properties of the evaluation subgroup of a map are established in [35, 36, 37]. As we mentioned earlier, Woo and Lee introduced the $G$-sequence of a map in [38]. Results on the $G$-sequence basically fall into one of three areas: conditions under which the $G$-sequence is exact (e.g. [38, Th.12] and [39]), examples of non-exactness (e.g. [14] and [24]), and extensions and generalizations of evaluation subgroups and the $G$-sequence (e.g. [15, 16]). These results give the stepping-off point for our work in this section. Example 4.1 demonstrates that the rationalized $G$-sequence may be non-exact at each of the three types of terms. Since non-exactness rationally implies non-exactness integrally, this example provides a new, complete example of non-exactness of the $G$-sequence. In contrast, Theorems 4.3 and 4.5 give general results concerning the exactness of the $G$-sequence. Our last development is a connection between the $G$-sequence of certain fibre inclusion maps and a well-known conjecture in rational homotopy theory (Theorem 4.9). Finally, we include an appendix, in which we give a careful and detailed justification of a technical fact about minimal models that is used in the proof of Theorem 2.1.

Before turning to our notational conventions, we make a few remarks on the background of our main result, Theorem 2.1, the identification of the rational homotopy groups of a function space in terms of derivations of the Sullivan model. We can identify three significant “tributary streams” that flow from rational homotopy theory into this work precisely at Theorem 2.1 at varying levels of generality. The first stream concerns the rationalized Gottlieb groups of a space. In [2], Félix and Halperin gave a characterization of the rationalized Gottlieb groups of a space, in terms of derivations of the Sullivan minimal model. At its most specialized level, Theorem 2.1 retrieves this characterization (Corollary 2.5) and extends it to a similar characterization of the rationalized evaluation subgroups of a map (Corollary 2.6). Félix and Halperin went on to prove a remarkable result concerning the rationalized Gottlieb groups of a finite complex [2, Th.III]. While no analogous such global results seem forthcoming for the rationalized evaluation subgroups of a map, our characterization of the rationalized evaluation subgroups is equally effective for concrete computations as is the earlier characterization of the rationalized Gottlieb groups.

The second tributary concerns $B \text{aut}_1(X)$—the classifying space for fibrations with fibre $X$ (we are denoting $\text{aut}_1(X)$ by $\text{map}(X, X; 1)$ here). The connection arises because we have an isomorphism of homotopy groups $\pi_{i+1}(B \text{aut}_1(X)) \cong \pi_i(\text{map}(X, X; 1))$. Also the Gottlieb groups of $X$ are obtained as the image in homotopy groups of the connecting homomorphism of the corresponding classifying fibration [6, Th.2.6]. Specializing to the case $f = 1_X$, Theorem 2.1 gives an identification of the map induced on rational homotopy groups by the evaluation map $\omega : \text{map}(X, X; 1) \rightarrow X$. In particular, it identifies the rational homotopy groups of $\text{map}(X, X; 1)$ in terms of derivations of the minimal model of $X$. This
last identification is familiar in rational homotopy theory. In [30, Sec.11], Sullivan sketched (with no proof) a model for the rational homotopy type of $B \text{aut}_1(X)$, from which the description of the rational homotopy groups of $\text{map}(X, X; 1)$ contained in Theorem 2.1 follows. A justification of Sullivan’s model for $B \text{aut}_1(X)$ may be gleaned by collating results from a number of sources spread through the literature (e.g. [27, 32, 31, 5]). However, a proof along these lines makes essential use of the role of the identity component in the classification of fibrations and cannot be extended to the general component $\text{map}(X, Y; f)$. Meier [20, (1.4), (2.6)] outlines the basic idea for a direct proof of the identification of the rational homotopy groups of $\text{map}(X, X; 1)$ but he is actually focussed on a special kind of situation in which the minimal model can be replaced by its cohomology and the argument given does not suffice for the general case. Félix and Thomas [4, Sec.2.3] state the description contained in Theorem 2.1 for the rational homotopy groups of $\text{map}(X, X; 1)$ but only a sketch of the proof is provided. Theorem 2.1 thus gives the first direct and detailed proof for the identification of the rational homotopy groups of $\text{map}(X, X; 1)$. Moreover, by extending this identification to the general component $\text{map}(X, Y; f)$, it provides a natural framework for the study of rational homotopy groups of function spaces.

Finally, the third tributary from rational homotopy theory consists of a model, described by Sullivan [30, Sec.11] with complete details provided by Haefliger [8] for the rational homotopy type of the components of the space of sections of a nilpotent fibration. By specializing to the trivial fibration $X \times Y \to X$, this yields a model for the function space $\text{map}(X, Y; f)$—and more generally a model for the rational homotopy type of the general evaluation map $\omega$. Now this model should in principle determine the rational homotopy groups of the function space (see [4] and [22], where it is used quite effectively). However, the model in question is a (non-minimal) DG algebra model. Therefore, the homomorphism induced by $\omega$ on rational homotopy groups—which is exactly the information we require to proceed with our development—is available only indirectly, at best. By focussing on the rational homotopy groups—as opposed to the rational homotopy type, we have arrived in Theorem 2.1 at an entirely new characterization of the map induced on rational homotopy groups by $\omega : \text{map}(X, Y; f) \to Y$. Furthermore, we have been able to give a direct proof that avoids many of the technical complexities of Haefliger’s work and is completely independent of it.

Our notation and terminology is that standard in the “simply connected of finite type” case of rational homotopy theory. Thus $X$ and $Y$ denote simply connected CW complexes of finite type. By vector space we mean a rational graded vector space. By algebra, we mean a commutative graded algebra over the rationals that is non-negatively graded, connected ($H^0 = \mathbb{Q}$), simply connected ($H^1 = 0$), and with cohomology of finite type. For a vector space $V$, we denote the free commutative graded algebra generated by $V$ by $\Lambda V$. We use the acronym DG to denote differential graded: Thus, DG vector space, DG algebra, and so-forth. For a DG algebra, the differential is of degree +1. In other situations, however, particularly when we consider the complex of derivations of a DG algebra, the differential is of degree –1. We will generally refer to a DG vector space whose differential is of degree –1 as a chain complex. If $f : A \to B$ is a map, either topological or algebraic, then $f^*$ denotes pre-composition by $f$ and $f_*$ denotes post-composition by $f$. In any setting in which it is appropriate, we use $H(f)$ to denote the map induced on homology.
(or cohomology) by \( f \), and \( f_\# \) to denote the map induced on homotopy groups by the map of spaces \( f \). A map of DG algebras is called a quasi-isomorphism if it induces an isomorphism on cohomology. We use \( \omega \) in a generic way to denote an evaluation map, and we denote the identity map of a topological space or the identity homomorphism of an algebra by 1. We denote the rationalization of a space \( X \) by \( X_\mathbb{Q} \) and of a map \( f \) by \( f_\mathbb{Q} \) (cf. [12]).

We assume that the reader is familiar with the basics of rational homotopy. Our general reference for this material is [3]. We recall here that a space \( X \) has a minimal model \( (M_X, d_X) \), with \( M_X \) a free algebra \( \mathbb{A} \) and \( d_X \) a decomposable differential, that is, \( d_X(V) \subseteq \Lambda^2 V \). Furthermore, a map of spaces \( f: X \to Y \) induces a map of minimal models \( M_f: M_Y \to M_X \). We refer to this induced map as the Sullivan minimal model of the map \( f \). It is a complete rational homotopy invariant for a map, and in principle all rational homotopy theoretic information about \( f \) can be retrieved from it. Passing to cohomology, for example, gives \( H(M_f): H^*(M_Y) \to H^*(M_X) \), which corresponds to the homomorphism of rational cohomology algebras induced by \( f \). The results of this paper illustrate how deeper information about a space or map may be retrieved from the minimal model by making correspondingly more sophisticated constructions with the model.

2. Derivation Spaces

Our purpose in this and the next section is to give a unified description in rational homotopy theory of all the terms involved in the definition of the \( G \)-sequence. Informally stated, we show that the homology theory of derivation complexes of Sullivan minimal models provides an algebraic model for the rational homotopy theory of function spaces at the level of homotopy groups.

We focus on the commutative square obtained from (1) by passing to homotopy groups. It turns out that identifying the rationalization of this commutative square allows us to conclude several results of interest.

We say two maps of vector spaces \( f: U \to V \) and \( g: U' \to V' \) are equivalent if there exist isomorphisms \( \alpha \) and \( \beta \) which make the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{f} & V \\
\downarrow{\alpha} & & \downarrow{\beta} \\
U' & \xrightarrow{g} & V'
\end{array}
\]

commutative. This notion of equivalence for vector space maps extends in the obvious way to sequences of vector space maps, commutative squares of vector space maps, and any other diagram of vector space maps.

Suppose a DG algebra \( (A, d_A) \) is isomorphic to \( \mathbb{Q} \) in degree zero, that is, \( A^0 \cong \mathbb{Q} \). Then the map \( \varepsilon: A \to \mathbb{Q} \) that sends all elements of positive degree to zero, and is the identity in degree zero, is the unique augmentation of \( A \). Here, as in the sequel, we regard \( \mathbb{Q} \) as the trivial DG algebra concentrated in degree zero and with trivial differential. Thus \( \varepsilon \) is a DG algebra map.

Given DG algebras \( (A, d_A) \) and \( (B, d_B) \) and a (fixed) DG algebra map \( \phi: A \to B \), define a \( \phi \)-derivation of degree \( n \) to be a linear map \( \theta: A \to B \) that reduces degree by \( n \) and satisfies the derivation law \( \theta(xy) = \theta(x)\phi(y) + (-1)^{|x|}\phi(x)\theta(y) \). We will only consider derivations of positive degree, that is, those that reduce degree by some
positive integer. Let $\text{Der}_n(A, B; \phi)$ denote the vector space of $\phi$-derivations of degree $n$, for $n > 0$. Finally, define a linear map $D: \text{Der}_n(A, B; \phi) \to \text{Der}_{n-1}(A, B; \phi)$ by $D(\theta) = d_B \circ \theta - (-1)^{\theta} d_A$. A standard check now shows that $D^2 = 0$ and thus $(\text{Der}_n(A, B; \phi), D)$ is a chain complex. In case $A = B$ and $\phi = 1_B$, the chain complex of derivations $\text{Der}_n(B, B; 1)$ is just the usual complex of derivations on the DG algebra $B$. In order to cut down on cumbersome notation, we will usually suppress the differential from our notation, and write $H_n(\text{Der}(A, B; \phi))$ for the homology in degree $n$ of the chain complex $(\text{Der}_n(A, B; \phi), D)$. Pre-composition with the DG algebra map $\phi: A \to B$ thus gives a map of chain complexes $\phi^* : \text{Der}(B, B; 1) \to \text{Der}(A, B; \phi)$. Furthermore, post-composition by the augmentation $\varepsilon : B \to Q$ induces a DG vector space map $\varepsilon_* : \text{Der}_*(A, B; \phi) \to \text{Der}_*(A, Q; \varepsilon)$.

All of the above can be applied to a map of minimal models. Suppose $f : X \to Y$ is a map of spaces, and $\mathcal{M}_f : \mathcal{M}_Y \to \mathcal{M}_X$ is the corresponding Sullivan minimal model of the map $f$. Then we have a commutative square of chain complexes

\[
\begin{array}{ccc}
\text{Der}_*(\mathcal{M}_X, \mathcal{M}_X; 1) & \xrightarrow{(\mathcal{M}_f)^*} & \text{Der}_*(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f) \\
\varepsilon_* & & \varepsilon_* \\
\text{Der}_*(\mathcal{M}_X, Q; \varepsilon) & \xrightarrow{(\mathcal{M}_f)^*} & \text{Der}_*(\mathcal{M}_Y, Q; \varepsilon)
\end{array}
\]

In this square, $\varepsilon : \mathcal{M}_X \to Q$ is the augmentation. Both horizontal maps are obtained by pre-composing with the same map $\mathcal{M}_f$, but in different contexts. Since we will need to distinguish between these two maps notationally in the sequel, we have used an extra decoration on the bottom one.

The main result of this section is the following:

**Theorem 2.1.** Let $X$ and $Y$ be simply connected CW complexes of finite type, with $X$ finite. For $n \geq 2$, the commutative square obtained by passing to rational homotopy groups from (1) is equivalent to the square obtained by passing to homology from (2). That is, the commutative squares

\[
\begin{array}{ccc}
\pi_n(\text{map}(X, X; 1)) \otimes Q & \xrightarrow{(f_*)_\omega \otimes Q} & \pi_n(\text{map}(X, Y; f)) \otimes Q \\
\omega_\omega \otimes Q & & \omega_\omega \otimes Q \\
\pi_n(X) \otimes Q & \xrightarrow{f_\omega \otimes Q} & \pi_n(Y) \otimes Q
\end{array}
\]

and

\[
\begin{array}{ccc}
H_n(\text{Der}(\mathcal{M}_X, \mathcal{M}_X; 1)) & \xrightarrow{H((\mathcal{M}_f)^*)} & H_n(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)) \\
H(\varepsilon_*) & & H(\varepsilon_*) \\
H_n(\text{Der}(\mathcal{M}_X, Q; \varepsilon)) & \xrightarrow{H((\mathcal{M}_f)^*)} & H_n(\text{Der}(\mathcal{M}_Y, Q; \varepsilon))
\end{array}
\]

are equivalent for each $n \geq 2$.

We prove this result below. First, we comment on some ingredients of the statement and proof, and give some immediate consequences.

**Remark 2.2.** In rational homotopy theory there is a standard way to identify the rational homotopy groups of a space $X$. Namely, the rational homotopy groups are identified with the dual of the vector space of indecomposables of the minimal
model, thus \( \pi_*(X) \otimes \mathbb{Q} \cong \text{Hom}(Q(M_X), \mathbb{Q}) \). More generally, \( f# \otimes \mathbb{Q} \) is identified with the dual of the map \( Q(M_f) : Q(M_Y) \to Q(M_X) \) induced by the Sullivan minimal model \( M_f : M_Y \to M_X \) (see [3, Sec.15(d)] for details). From the bottom maps in the two squares of Theorem 2.1, we obtain a superficially different description of \( f# \otimes \mathbb{Q} \). But it is easy to see that this agrees with the standard one: Note that the derivation law implies \( \text{Der}_*(M_X, \mathbb{Q}; \varepsilon) \cong \text{Hom}(Q(M_X), \mathbb{Q}) \), while the minimality of \( M_X \) implies that \( D = 0 \) in the chain complex \( \text{Der}_*(M_X, \mathbb{Q}; \varepsilon) \). Thus we have \( \text{Hom}(Q(M_X), \mathbb{Q}) \cong H_*(\text{Der}(M_X, \mathbb{Q}; \varepsilon)) \). The agreement between our description of \( f# \otimes \mathbb{Q} \) and the standard one is obvious from this isomorphism.

Theorem 2.1 also contains and depends upon basic results concerning the rationalization of function space components and evaluation subgroups due to several authors. We consider this material here.

When \( X \) is a finite complex, the function space map \( (X,Y) \) has the homotopy type of a CW complex by the result of Milnor [21]. In fact, by Hilton-Mislin-Roitberg [12, Th.II.2.5] the components map \( (X,Y;f) \) are nilpotent complexes. Moreover, given a rationalization \( e_Y : Y \to Y_{\mathbb{Q}} \) of \( Y \) the induced map \( (e_Y)_* : \text{map}(X,Y;f) \to \text{map}(X,Y_{\mathbb{Q}};e_Y \circ f) \) is a rationalization [12, Th.II.3.11]. By [28, Th.2.3], rationalization in the initial variable \( e_X : X \to X_{\mathbb{Q}} \) induces a weak equivalence \( (e_X)^*: \text{map}(X_{\mathbb{Q}}, Y_{\mathbb{Q}};f_{\mathbb{Q}}) \to \text{map}(X,Y_{\mathbb{Q}};f_{\mathbb{Q}} \circ e_X) \). These results, together with the naturality of the various maps involved, imply the following result.

**Theorem 2.3.** Let \( f : X \to Y \) be a map between simply connected complexes of finite type with \( X \) finite. Let \( f_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}} \) denote the rationalization of \( f \). The rational homotopy ladder obtained from (1) is equivalent to the homotopy ladder obtained from the following commutative square:

\[
\begin{array}{ccc}
\text{map}(X_{\mathbb{Q}}, Y_{\mathbb{Q}};1) & \xrightarrow{(f_{\mathbb{Q}})_*} & \text{map}(X_{\mathbb{Q}}, Y_{\mathbb{Q}};f_{\mathbb{Q}}) \\
\omega \downarrow & & \omega \\
X_{\mathbb{Q}} & \xrightarrow{f_{\mathbb{Q}}} & Y_{\mathbb{Q}}
\end{array}
\]

In particular, the commutative square (3) and the commutative square obtained by passing to \( n \)th homotopy groups from (4) are equivalent for each \( n \geq 2 \).

**Proof.** Use the rationalizations \( ((e_Y)_*, (e_X)_*, e_Y, e_X) \) and the weak equivalence \( ((e_X)^*) \) mentioned above to construct a map from the first (ordinary) homotopy ladder to the second. Repeated use of the five-lemma shows that all maps from one ladder to the other are rationalization homomorphisms. \( \square \)

The result \( G_n(X_{\mathbb{Q}}) \cong G_n(X) \otimes \mathbb{Q} \) for \( X \) a simply connected finite complex due to Lang [13] is an easy consequence of Theorem 2.3. But we can say more. Because of the naturality of the maps involved, the rationalizations in the proof between the two ladders induce rationalization homomorphisms of the respective image sequences. Therefore, we obtain the following:

**Corollary 2.4.** Let \( f : X \to Y \) be a map between simply connected complexes of finite type with \( X \) finite. Then the rationalization of the \( G \)-sequence of \( f \)

\[
\cdots \to G_n(X) \otimes \mathbb{Q} \to G_n(Y,X;f) \otimes \mathbb{Q} \to G_{n-1}^{rel}(Y,X;f) \otimes \mathbb{Q} \to G_{n-2}(X) \otimes \mathbb{Q} \to \cdots
\]
is equivalent to the $G$-sequence of the rationalization of $f$

\[ \cdots \to G_n(X_Q) \to G_n(Y_Q, X_Q; f_Q) \to G_n^{rel}(Y_Q, X_Q; f_Q) \to G_{n-1}(X_Q) \to \cdots \]

In particular, we have isomorphisms $G_n(Y_Q, X_Q; f_Q) \cong G_n(Y, X; f) \otimes \mathbb{Q}$ (cf. [36, 28]) and $G_n^{rel}(Y_Q, X_Q; f_Q) \cong G_n^{rel}(Y, X; f) \otimes \mathbb{Q}$ for $n \geq 2$.

In the next section, we identify the rational $G$-sequence in the context of minimal models. Meanwhile, from Theorem 2.1 we immediately retrieve minimal model descriptions of the rationalized Gottlieb groups of a space, and of the rationalized evaluation subgroups of a map. The first of these is well-known:

**Corollary 2.5.** Let $X$ be a simply connected finite complex. The rationalized $n$th Gottlieb group $G_n(X_Q) \cong G_n(X) \otimes \mathbb{Q}$ is isomorphic to the image of the induced homomorphism $H(\varepsilon_*) : H_n(Der(\mathcal{M}_X, \mathcal{M}_X; 1)) \to H_n(Der(\mathcal{M}_X, \mathcal{M}_X; \mathbb{Q}; \varepsilon))$ for $n \geq 2$.

This is easily translated into the standard minimal model description of the rationalized Gottlieb groups given by Félix and Halperin (see [3, Sec.29(c)]). Specifically, they describe a Gottlieb element of the minimal model $\mathcal{M}_X = \Lambda(V)$ as a linear map $\theta : V^n \to \mathbb{Q}$ that extends to a derivation of $\mathcal{M}_X$ satisfying $d_\theta = (-1)^n d\theta$. Such a derivation $\theta$ is a cycle in $\text{Der}(\mathcal{M}_X, \mathcal{M}_X; 1)$, and the class that it represents has non-zero image under $H(\varepsilon_*)$ precisely when the original linear map $\theta : V^n \to \mathbb{Q}$ is non-zero. On recalling that $H_n(Der(\mathcal{M}_X, \mathcal{M}_X; \mathbb{Q}; \varepsilon)) \cong \text{Der}(\mathcal{M}_X, \mathcal{M}_X; \mathcal{M}_f) \cong \text{Hom}(\mathbb{Q}(\mathcal{M}_X), \mathbb{Q})$, we see the two descriptions agree.

**Corollary 2.6.** Let $f : X \to Y$ be a map between simply connected complexes of finite type with $X$ finite. The rationalized $n$th evaluation subgroup $G_n(Y_Q, X_Q; f_Q) \cong G_n(Y, X; f) \otimes \mathbb{Q}$ of the map $f$ is isomorphic to the image of the induced homomorphism $H(\varepsilon_*) : H_n(Der(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)) \to H_n(Der(\mathcal{M}_Y, \mathcal{M}_Y; \mathbb{Q}; \varepsilon))$ for $n \geq 2$.

This identification of the rationalized evaluation subgroups of the map $f$ can be conveniently phrased in a way comparable to the Félix-Halperin description of the rationalized Gottlieb groups: An evaluation subgroup element of the minimal model $\mathcal{M}_f : \mathcal{M}_Y \to \mathcal{M}_X$, with $\mathcal{M}_Y = \Lambda(W)$, is a linear map $\theta : W^n \to \mathbb{Q}$ that extends to an $\mathcal{M}_f$-derivation $\theta : \mathcal{M}_Y \to \mathcal{M}_X$ satisfying $d_\theta = (-1)^n d\theta$.

**Definition 2.7.** Suppose $\phi : A \to B$ is a map of DG algebras. We define the evaluation subgroup of $\phi$ as the image of the map

\[ H(\varepsilon_*) : H_n(Der(A, B; \phi)) \to H_n(Der(A, B; \phi)). \]

We denote it by $G_n(A, B; \phi)$. In the special case in which $A = B$ and $\phi = 1_B$, we refer to the Gottlieb group of $B$, and use the notation $G_n(B)$.

From the previous discussion, we see that $G_n(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f) \cong G_n(Y, X; f) \otimes \mathbb{Q}$ and $G_n(\mathcal{M}_X) \cong G_n(X) \otimes \mathbb{Q}$.
Proof of Theorem 2.1. We will define vector space isomorphisms $\Phi$, $\Phi_f$, $\Psi_X$, and $\Psi_Y$ to give the following equivalence of commutative squares:

$$
\begin{array}{cccc}
H_n(\text{Der}(\mathcal{M}_X, \mathcal{M}_X; 1)) & \xrightarrow{H(f, f^*)} & H_n(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)) \\
\pi_n(\text{map}(X, X; 1)) \otimes \mathbb{Q} & \xrightarrow{(f_*)\otimes 1} & \pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q} & \xrightarrow{H(f, f^*)} & H_n(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)) \\
\pi_n(X) \otimes \mathbb{Q} & \xrightarrow{\omega \cdot f \otimes 1} & \pi_n(Y) \otimes \mathbb{Q} & \xrightarrow{\omega \cdot H(f, f^*)} & H_n(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)) \\
\end{array}
$$

We obtain $\Phi_f$ as the rationalization of a natural homomorphism

$$
\Phi_f': \pi_n(\text{map}(X, Y; f)) \rightarrow H_n(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f))
$$

which we now define. A representative of a homotopy class $\alpha \in \pi_n(\text{map}(X, Y; f))$ determines, via the exponential correspondence, a map $F: S^n \times X \rightarrow Y$ that satisfies $F \circ i_2 = f$, where $i_2: X \rightarrow S^n \times X$ is the inclusion. The map $F$ is often called the adjoint map of (a representative of) $\alpha$. Passing to minimal models, we obtain a map $\mathcal{M}_F: \mathcal{M}_Y \rightarrow \mathcal{M}_{S^n} \otimes \mathcal{M}_X$ with $\mathcal{M}_i \circ \mathcal{M}_F = \mathcal{M}_f$ (equals, not just up to DG homotopy—see Proposition A.2 of the appendix). Now $S^n$ is a formal space, which means there is a quasi-isomorphism of DG algebras $\psi: \mathcal{M}_{S^n} \rightarrow H^*(S^n; \mathbb{Q})$. In turn, this gives a quasi-isomorphism $\psi \otimes 1: \mathcal{M}_{S^n} \otimes \mathcal{M}_X \rightarrow H^*(S^n; \mathbb{Q}) \otimes \mathcal{M}_X$. Recall that $AV$ denotes the free algebra generated by $V$. That is, polynomial on generators of even degree, and exterior on generators of odd degree. Write $H^*(S^n; \mathbb{Q})$ as $\Lambda(s_n)/(s_n^2)$ if $n$ is even, or as $\Lambda(s_n)$ if $n$ is odd. Given $\chi \in \mathcal{M}_Y$, we may write

$$
(\psi \otimes 1) \circ \mathcal{M}_F(\chi) = 1 \otimes \mathcal{M}_f(\chi) + s_n \otimes \theta_F(\chi),
$$

thus defining a linear map $\theta_F: \mathcal{M}_Y \rightarrow \mathcal{M}_X$ that reduces degree by $n$. A standard check—using the fact that $(\psi \otimes 1) \circ \mathcal{M}_F$ is a DG algebra map—shows that $\theta_F$ is an $\mathcal{M}_f$-derivation that is a $D_{\mathcal{M}_f}$-cycle. Set $\Phi_f'(\alpha) = [\theta_F] \in H_n(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)).$

To show $\Phi_f'(\alpha)$ is well-defined, suppose that $g_1, g_2: S^n \rightarrow \text{map}(X, Y; f)$ are homotopic representatives of $\alpha$ with adjoint maps $F, G: S^n \times X \rightarrow Y$ respectively. Then the homotopy $K: S^n \times I \rightarrow \text{map}(X, Y; f)$ from $g_1$ to $g_2$ gives a homotopy $H: S^n \times X \times I \rightarrow Y$, from $F$ to $G$, by setting $H(s, x, t) = K(s, t)(x)$. Further, since $K$ is a based homotopy, the homotopy $H$ satisfies $H \circ i = J$, where $i$ denotes the inclusion $i(x, t) = (s, x, t)$ and $J$ denotes the homotopy $J(x, t) = f(x)$ that is stationary at $f$. A basic result of rational homotopy theory says that homotopic maps have DG homotopic Sullivan minimal models (see [3, Ch.12]). So the homotopy $H$ gives a DG homotopy $\mathcal{H}: \mathcal{M}_Y \rightarrow \mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt)$ between minimal models for $F$ and $G$. Translating the restriction on $H$ into minimal model terms allows us to assume that $\mathcal{H}$ is such that

$$
(5) \quad (\psi \otimes 1) \circ \mathcal{H}(\chi) = 1 \otimes \mathcal{M}_f(\chi) \otimes 1 + \sum_{i \geq 0} s_n \otimes \alpha_i(\chi) \otimes t^i + \sum_{i \geq 0} s_n \otimes \beta_i(\chi) \otimes t^i dt
$$
for an element $\chi \in \mathcal{M}_Y$. This translation is intuitively plausible, but its justification requires some technical details, which we provide in Proposition A.2. Since the DG homotopy $\mathcal{H}$ is from $\mathcal{M}_F$ to $\mathcal{M}_G$, then at $t = 0$ we have $\alpha_0(\chi) = \theta_F(\chi)$, and from $t = 1$, we have $\sum_{i \geq 0} \alpha_i(\chi) = \theta_G(\chi)$. To establish well-definedness, we must show these differ by a boundary in $\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)$. To this end, use (5) to write separate expressions for $(\psi \otimes 1 \circ 1) \circ \mathcal{H}(\chi')$ and $(\psi \otimes 1 \circ 1) \circ \mathcal{H}(\chi')$, and these expressions agree. By equating them and collecting like terms we obtain equations

$$
\beta_i(\chi') = (\chi^1)^{n+1} \beta_i(\chi') + (\chi^2)^{n+1} \beta_i(\chi')
$$

for each $i \geq 0$. By substituting $\gamma_i(\chi) = (\chi^1)^{n+1} \beta_i(\chi)$ for $\chi \in \mathcal{M}_Y$, we obtain derivations $\gamma_i \in \text{Der}_{n+1}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)$. On the other hand, use (5) to write separate expressions for $(\psi \otimes 1 \circ 1) \circ \mathcal{H}(d\chi)$ and $d(\psi \otimes 1 \circ 1) \circ \mathcal{H}(\chi)$, with the latter obtained by applying $d$ to both sides of (5). Since a DG homotopy respects differentials, these expressions agree. By equating them and collecting like terms we obtain equations

$$
\alpha_{i+1}(\chi) = (\chi^1)^n \beta_i(\chi) + (\chi^2)^n \beta_i(\chi)
$$

for each $i \geq 0$. With the previous substitution, this gives $d(\gamma_i(\chi) = (\chi^1)^{n+1} \gamma_i d(\chi) = (\chi^2)^{n+1} \gamma_i d(\chi)$, that is,

$$
\alpha_{i+1}(\chi) = (\chi^1)^{n+1} \frac{1}{i+1} d_{\mathcal{M}_f}(\gamma_i(\chi))
$$

for each $i \geq 0$. It follows that the difference of derivations $\theta_G - \theta_F = \sum_{i \geq 1} \alpha_i$ is a $D_{\mathcal{M}_f}$-boundary in $\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)$. Hence $\Phi'_f$ is well-defined.

Next we show that $\Phi'_f$ is a homomorphism. Suppose $\alpha, \beta \in \pi_n(\text{map}(X, Y; f))$ have adjoints $A, B : S^n \times X \to Y$. Let $\sigma : S^n \to S^n \vee S^n$ denote the usual pinching comultiplication. Then the sum $\gamma := \alpha + \beta$ is the composition $(\alpha | \beta) \circ \sigma$. Let $i_1, i_2 : S^n \to S^n \vee S^n$ denote the inclusions, and let $(A | B)_f : (S^n \vee S^n) \times X \to Y$ be the map defined by $(A | B)_f \circ (i_1 \times 1) = A$ and $(A | B)_f \circ (i_2 \times 1) = B$. Then the adjoint of $\gamma$ is $C := (A | B)_f \circ (\sigma \times 1) : S^n \times X \to Y$. We focus on identifying a Sullivan minimal model of $C$, and it will follow that $\Phi'_f$ is a homomorphism.

First note that the comultiplication $\sigma$ is a formal map. In fact, let $\psi : \mathcal{M}_{S^n} \to H^*(S^n; \mathbb{Q})$ be the quasi-isomorphism above, and let $\psi' : \mathcal{M}_{S^n \vee S^n} \to H^*(S^n \vee S^n; \mathbb{Q})$ be a quasi-isomorphism for the formal space $S^n \vee S^n$. To say $\sigma$ is formal is to say that the compositions $\psi \circ \sigma$ and $H(\sigma) \circ \psi'$ are DG homotopic. Indeed, here, it is evident that we may choose $\psi'$ so that $\psi \circ \sigma = H(\sigma) \circ \psi'$. Then we may identify $(\psi' \circ 1) \circ \mathcal{M}_C$, which is the model we seek, with $(H(\sigma) \circ 1) \circ \psi' \circ 1 \circ \mathcal{M}_{(A|B)_f}$.

Let $\Gamma : \mathcal{M}_Y \to H^*(S^n \vee S^n; \mathbb{Q}) \otimes \mathcal{M}_X$ denote the composition $(\psi' \circ 1) \circ \mathcal{M}_{(A|B)_f}$. Without loss of generality (cf. Proposition A.2) we may assume that $\Gamma$ is of the form

$$
\Gamma(\chi) = 1 \otimes \mathcal{M}_f(\chi) + s_n \otimes \theta_1(\chi) + t_n \otimes \theta_2(\chi),
$$

for $\chi \in \mathcal{M}_Y$. Here we have written $H^*(S^n \vee S^n; \mathbb{Q})$ as $\Lambda(s_n, t_n)/(s_n^2, t_n^2, s_n t_n)$ if $n$ is even, or as $\Lambda(s_n, t_n)/(s_n t_n)$ if $n$ is odd. Furthermore, $\theta_1$ and $\theta_2$ are cycles in $\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)$, for the same reason that the general $\theta_F$ is a cycle in the first part of this proof above. Because $(A | B)_f \circ (i_1 \times 1) = A$, this model gives a map $\Gamma_1(\chi) = 1 \otimes \mathcal{M}_f(\chi) + s_n \otimes \theta_1(\chi)$ that is DG homotopic to $\mathcal{M}_A$. Furthermore, the DG homotopy here may be assumed to be restricted in the way described in Proposition A.2. Suppose $A$ has minimal model such that $(\psi \circ 1) \circ \mathcal{M}_A(\chi) = \ldots$
1 \otimes \mathcal{M}_f(\chi) + s_n \otimes \theta_A(\chi)$. As in the previous part of this proof, it follows that $[\theta_1] = [\theta_A]$ in $H_n(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f))$. Likewise, we have $[\theta_2] = [\theta_B]$, where $B$ has minimal model such that $(\psi \otimes 1) \circ \mathcal{M}_B(\chi) = 1 \otimes \mathcal{M}_f(\chi) + s_n \otimes \theta_B(\chi)$. Finally, since $\sigma : S^n \to S^n \vee S^n$ induces $H(\sigma) : H^*(S^n \vee S^n; \mathbb{Q}) \to H^*(S^n; \mathbb{Q})$ given by $H(\sigma)(s_n) = s_n$ and $H(\sigma)(1) = s_n$, we have

\[ (H(\sigma) \otimes 1) \circ (\psi' \otimes 1) \circ \mathcal{M}_{(A|B)}(\chi) = 1 \otimes \mathcal{M}_f(\chi) + s_n \otimes (\theta_1(\chi) + \theta_2(\chi)). \]

Up to a further DG homotopy, once again restricted in the way described in Proposition A.2, this is $(\psi \otimes 1) \circ \mathcal{M}_C$. Hence $[\theta_C] = [\theta_1] + [\theta_2] = [\theta_A] + [\theta_B]$ and $\Phi'_f$ is a homomorphism.

As stated before, the map of vector spaces $\Phi_f$ is now obtained as the rationalization of the (group) homomorphism $\Phi'_f$. The map $\Phi$ is defined in the same way, specializing to the case in which $Y = X$ and $f = 1_X$. The maps $\Phi_X$ and $\Phi_Y$ are the standard minimal model identification of the rational homotopy groups of a space, as discussed in Remark 2.2.

Next, we show $\Phi_f$ is surjective. Denote by $[S^n_Q \times X, Y_Q]_{f_Q}$ the subset of the set of homotopy classes of maps $S^n_Q \times X \to Y_Q$ consisting of classes represented by a map that restricts to $f_Q$ on $X_Q$. By Theorem 2.3, we identify $\pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q}$ with $\pi_n(\text{map}(X_Q, Y_Q; f_Q))$, and hence with $[S^n_Q \times X, Y_Q]_{f_Q}$. Now suppose given $[\theta] \in H_n(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f))$. Use $\theta$ to define

\[ \phi(\chi) = 1 \otimes \mathcal{M}_f(\chi) + s_n \otimes \theta(\chi) \]

for $\chi \in \mathcal{M}_Y$. Since $\theta$ is an $\mathcal{M}_f$-derivation that is a cycle, this defines a DG algebra map $\phi : \mathcal{M}_Y \to H^*(S^n_Q) \otimes \mathcal{M}_X$. Now lift $\phi$ through the surjective quasi-isomorphism $\psi \otimes 1$ as in [3, Lem.12.4], to obtain a map $\tilde{\phi} : \mathcal{M}_Y \to \mathcal{M}_S = \mathcal{M}_X$ that satisfies $(\varepsilon \cdot 1) \circ \tilde{\phi} = \mathcal{M}_f : \mathcal{M}_Y \to \mathcal{M}_X$. By the standard correspondence between maps of minimal models and maps of rational spaces, this gives a map $F : S^n_Q \times X_Q \to Y_Q$ that satisfies $i_2 \circ F \sim f_Q : X_Q \to Y_Q$. Using, for example, [3, Th.9.7], we can adjust $F$ into a homotopic map $F' : S^n_Q \times X_Q \to Y_Q$ that satisfies $i_2 \circ F' = f_Q : X_Q \to Y_Q$, so that $F'$ represents a class of $[S^n_Q \times X, Y_Q]_{f_Q}$. As described at the start of this paragraph, $F'$ corresponds to a homotopy class $\alpha \in \pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q}$. Evidently, we have $\Phi_f(\alpha) = \theta$.

Finally, we show $\Phi_f$ is injective. Since $\Phi_f$ is a vector space homomorphism, it is sufficient to show that $\alpha \in \pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q}$ is zero whenever $\Phi_f(\alpha) = 0$. Using the identification of the previous paragraph, let $G : S^n_Q \times X_Q \to Y_Q$ be the adjoint map for $\alpha$. Suppose that $\theta_G = D(\eta)$ for $\eta \in \text{Der}_{n+1}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)$. Using $\eta$, define a map $\Gamma : \mathcal{M}_Y \to H^*(S^n_Q) \otimes \mathcal{M}_X \otimes \Lambda(t, dt)$ by

\[ \Gamma(\chi) = 1 \otimes \mathcal{M}_f(\chi) \otimes 1 + s_n \otimes \theta_G(\chi) \otimes (1 - t) + s_n \otimes \eta(\chi) \otimes dt. \]

A routine check verifies that $\Gamma$ is a DG algebra map. Furthermore, it is a DG homotopy from $\mathcal{M}_G$ to the map $E : \mathcal{M}_Y \to H^*(S^n_Q) \otimes \mathcal{M}_X$ given by $E(\chi) = 1 \otimes \mathcal{M}_f(\chi)$. Now this latter map is a Sullivan model of the adjoint of $0 \in \pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q}$. Therefore, the DG homotopy translates into a homotopy between adjoint maps $S^n_Q \times X_Q \to Y_Q$ for $\alpha$ and $0$. It follows that $\alpha = 0$, and thus $\Phi_f$ is injective. We observe that, strictly speaking, we have not justified that the homotopy between the maps $S^n_Q \times X_Q \to Y_Q$ is relative to $X_Q$, which corresponds to the homotopy between the maps $S^n \to \text{map}(X_Q, Y_Q; f_Q)$ being based. However, a based map from a sphere is null-homotopic if and only if it is based null-homotopic (cf. [29, p.27]).
Commutativity of the cube diagram follows from the naturality of the homomorphism \( \Phi_f \). By naturality, we mean the following: Suppose given maps of spaces \( f: A \to B \) and \( g: B \to C \). The we have induced maps of function spaces \( g_*: \text{map}(A,B; f) \to \text{map}(A,C; g \circ f) \) and \( f^*: \text{map}(B,C; g) \to \text{map}(A,C; g \circ f) \). For either case we obtain a commutative square involving \( \Phi_{g \circ f} \). Namely, we have \( H((\mathcal{M}_g)^*) \circ \Phi_f = \Phi_{g \circ f} \circ (g_*) \# \) in the first case, and \( H((\mathcal{M}_f)_*) \circ \Phi_g = \Phi_{g \circ f} \circ (f^*) \# \) in the second case, as is easily checked. Since \( \Phi_f \) is obtained from \( \Phi_f \) by localization, the isomorphism \( \Phi_f \) has the same naturality property. This is sufficient to conclude that the top, bottom, left, and right faces of the cube commute. For the evaluation map \( \omega: \text{map}(X,Y; f) \to Y \) can be identified with \( i^*: \text{map}(X,Y; f) \to \text{map}(x_0, Y; y_0) \), where \( x_0 \in X \) and \( y_0 \in Y \) denote basepoints, and \( i: x_0 \to X \) inclusion of the basepoint. Likewise for the evaluation map \( \omega: \text{map}(X,Y; f) \to X \), and then \( f: X \to Y \) can be identified with \( f_*: \text{map}(x_0, X; x_0) \to \text{map}(x_0, Y; y_0) \) (cf. also Remark 2.2). Finally, the front and rear faces commute because the squares (1) and (2) are commutative. 

We finish this section with some immediate consequences of Theorem 2.1. The first retrieves a basic result of Thom, in the rational homotopy setting.

**Corollary 2.8.** ([33, Th.2]) Let \( Y = K(V, m) \) be an Eilenberg-Mac Lane space, for \( V \) a finite dimensional (ungraded) rational vector space. If \( X \) is a finite CW complex, and \( f: X \to Y \) is any map, then \( \pi_n(\text{map}(X,Y; f)) \cong H^{m-n}(X; V) \).

**Proof.** The minimal model for \( Y \) is \( AV^* \) with zero differential, where \( V^* \) denotes the dual vector space of \( V \). It follows easily that \( H_n(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)) \cong \text{Hom}(V^*, H^{m-n}(X; \mathbb{Q})) \cong H^{m-n}(X; V) \). 

Notice that this result—with the remarks on rationalization preceding Theorem 2.3—easily extends to yield the rational homotopy type of \( (X,Y; f) \), in case \( Y \) is a rational \( H \)-space, that is, a space whose rationalization is an \( H \)-space.

We also obtain an easy proof of the following general result:

**Corollary 2.9.** Let \( X \) and \( Y \) be simply connected CW complexes with \( X \) finite and \( Y \) of finite type. Let \( f: X \to Y \) be any map. Then

\[
\dim (\pi_n(\text{map}(X,Y; f)) \otimes \mathbb{Q}) \leq \dim (\pi_n(\text{map}(X,Y; 0)) \otimes \mathbb{Q})
\]

for all \( n \geq 2 \).

**Proof.** The inequality follows from Theorem 2.1 and the inequality

\[
\dim (H_n(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f))) \leq \dim (H_n(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; 0)))
\]

In [19, Th.6.1], we establish the corresponding inequality for the rank of the fundamental groups.

We give a further consequence of Theorem 2.1 concerning the rational homotopy groups of certain function spaces. Define an \( F_0 \)-space to be a finite simply connected complex with finite dimensional rational homotopy (a rationally elliptic space) such that \( H^{odd}(X, \mathbb{Q}) = 0 \). This type of space features in the following well-known conjecture of Halperin (cf. [3, p.516]):

**Conjecture 2.10.** Suppose \( X \) is an \( F_0 \)-space. Then any fibration \( X \to E \to B \) of simply connected spaces is TNCZ, that is, the fibre inclusion \( j: X \to E \) induces a surjection on rational cohomology.
Corollary 2.11. Let \( f: X \to Y \) be a map between \( F_0 \)-spaces. Then for \( r \geq 1 \),
\[
\pi_{2r}(\text{map}(X, Y; f)) \otimes \mathbb{Q} \cong \text{Der}_{2r}(H^*(Y, \mathbb{Q}), H^*(X, \mathbb{Q}); H(f)).
\]

Proof. The argument given by Grivel for the case in which \( Y = X \) and \( f = 1_X \) can be directly adjusted to show
\[
H_{2r}(\text{Der}(M_Y, M_X; M_f)) \cong \text{Der}_{2r}(H^*(Y, \mathbb{Q}), H^*(X, \mathbb{Q}); H(f)).
\]
The result now follows from Theorem 2.1.

3. DERIVATION SPACES AND LONG EXACT SEQUENCES

In this section, we identify the rationalized long exact homotopy sequences of the
maps \( f: X \to Y \) and \( f_*: \text{map}(X, X; 1) \to \text{map}(X, Y; f) \), and hence the rationalized
\( G \)-sequence of \( f \). Our identifications flow from the observation that the third term
in a long exact sequence of vector spaces is unique up to (non-natural) isomorphism:

Lemma 3.1. Suppose given diagrams of vector spaces
\[
\begin{array}{ccc}
A_{n+1} & \xrightarrow{j_{n+1}} & C_{n+1} & \xrightarrow{k_{n+1}} & A_n & \xrightarrow{i_n} & B_n \\
\alpha_{n+1} & \xrightarrow{\beta_{n+1}} & \gamma_{n+1} & \xrightarrow{\alpha_n} & \beta_n \\
X_{n+1} & \xrightarrow{p_{n+1}} & Y_{n+1} & \xrightarrow{q_{n+1}} & Z_{n+1} & \xrightarrow{r_{n+1}} & Y_{n+1}
\end{array}
\]
for each \( n \geq 2 \). Suppose the rows are exact, each \( \alpha_n \) and \( \beta_n \) is an isomorphism, and
\( \beta_n \circ i_n = p_n \circ \alpha_n \) for each \( n \). Then there exist isomorphisms \( \gamma_{n+1}: C_{n+1} \to Z_{n+1} \),
for \( n \geq 2 \), which make the entire ladder commutative.

Proof. From the given exactness and commutativity properties, each \( \alpha_n \) restricts
to an isomorphism \( \alpha_n: \text{im}(k_{n+1}) \to \text{im}(r_{n+1}) \). Now consider the following diagram
of short exact sequences:
\[
\begin{array}{ccc}
0 & \xrightarrow{} & \text{im}(j_{n+1}) & \xrightarrow{} & C_{n+1} & \xrightarrow{\sigma_n} & \text{im}(k_{n+1}) & \xrightarrow{} & 0 \\
\gamma_{n+1} & \xrightarrow{} & 0 & \xrightarrow{} & \text{im}(q_{n+1}) & \xrightarrow{} & Z_{n+1} & \xrightarrow{\sigma_n} & \text{im}(r_{n+1}) & \xrightarrow{} & 0
\end{array}
\]
Choose splittings \( s_n \) and \( \sigma_n \) as indicated, so that \( C_{n+1} = \text{im}(j_{n+1}) \oplus s_n(\text{im}(k_{n+1})) \).
On \( \text{im}(j_{n+1}) \subseteq C_{n+1} \), define \( \gamma_{n+1}(j_{n+1}(b)) = q_{n+1} \circ \beta_{n+1}(b) \). On \( s_n(\text{im}(k_{n+1})) \),
define \( \gamma_{n+1}(s_n(a)) = \sigma_n \circ \alpha_n(a) \). It is easy to check \( \gamma_{n+1}: C_{n+1} \to Z_{n+1} \) is a
well-defined isomorphism that makes the original ladder commute.

This observation is useful since it allows us to choose descriptions of certain
long exact sequences that are most convenient for our purposes. We begin by
describing the long exact rational homotopy sequence of the map \( f_*: \text{map}(X, X; 1)
\to \text{map}(X, Y; f) \). For this purpose, we recall the definition of the mapping cone of
a chain map \( \phi: A \to B \) and use this construction to define the long exact homology
sequence of \( \phi \).
Definition 3.2. [29, p.166] Let \( \phi : A \to B \) be map of DG vector spaces. Define a DG vector space, called the mapping cone of \( \phi \) and denoted by \( \text{Rel}_*(\phi) \), as follows: \( \text{Rel}_n(\phi) = A_{n-1} \oplus B_n \), with differential \( \delta (= \delta_\phi) \) of degree \(-1\) given by \( \delta(a,b) = (-d_A(a), \phi(a) + d_B(b)) \). Further, define chain maps \( J : B_n \to \text{Rel}_n(\phi) \) and \( P : \text{Rel}_n(\phi) \to A_{n-1} \) by \( J(b) = (0, b) \) and \( P(a,b) = a \). These give a short exact sequence of chain complexes

\[
0 \to B_n \xrightarrow{J} \text{Rel}_n(\phi) \xrightarrow{P} A_{n-1} \to 0
\]

which leads to a long exact sequence in homology

\[
\cdots \to H_{n+1}(\text{Rel}(\phi)) \xrightarrow{H(P)} H_\alpha(A) \xrightarrow{H(\phi)} H_\alpha(B) \xrightarrow{H(J)} H_\alpha(\text{Rel}(\phi)) \to \cdots,
\]

whose connecting homomorphism is \( H(\phi) \). We refer to this sequence as the long exact homology sequence of \( \phi \).

Suppose given a commutative square of DG vector spaces

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
\phi \downarrow & & \beta \\
A' & \xrightarrow{\phi'} & B'.
\end{array}
\]

Then the obvious map \( (\alpha, \beta) : \text{Rel}_*(\phi) \to \text{Rel}_*(\phi') \) is a chain map that satisfies \((\alpha, \beta) \circ J = J' \circ \beta \) and \( \alpha \circ P = P' \circ (\alpha, \beta) \). Thus we obtain a homology ladder

\[
\cdots \to H_{n+1}(\text{Rel}(\phi)) \xrightarrow{H(P)} H_\alpha(A) \xrightarrow{H(\phi)} H_\alpha(B) \xrightarrow{H(J)} H_\alpha(\text{Rel}(\phi)) \to \cdots
\]

In particular, we can apply this construction to the map of chain complexes

\[
(\mathcal{M}_f)^* : \text{Der}_*(\mathcal{M}_X, \mathcal{M}_X; 1) \to \text{Der}_*(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)
\]

induced by the minimal model \( \mathcal{M}_f : \mathcal{M}_Y \to \mathcal{M}_X \) of the map \( f : X \to Y \).

Theorem 3.3. The long exact sequence induced by

\[
f_* : \text{map}(X, X; 1) \to \text{map}(X, Y; f)
\]

on rational homotopy groups is equivalent to the long exact homology sequence of the map (7). Specifically, this is a long exact sequence

\[
\cdots \xrightarrow{H(P)} H_{n+1}(\text{Rel}((\mathcal{M}_f)^*)) \xrightarrow{H(J)} H_n(\text{Der}(\mathcal{M}_X, \mathcal{M}_X; 1)) \xrightarrow{H(\mathcal{M}_f)^*} H_n(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)) \xrightarrow{H(J)} \cdots
\]

that we terminate in \( H_2(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)) \), in which \( \text{Rel}_*((\mathcal{M}_f)^*) \) is the relative chain complex of the map (7), as defined in Definition 3.2.
Proof. Theorem 2.1 gives equivalences of vector space maps

$$\pi_n(\text{map}(X, X; 1)) \otimes \mathbb{Q} \xrightarrow{(f_\ast)_\# \otimes 1} \pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q}$$

\[
\phi_1 \cong \phi_j
\]

\[
H_n(\text{Der}(\mathcal{M}_X, \mathcal{M}_X; 1)) \xrightarrow{H((\mathcal{M}_f)^* \nu)} H_n(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f)),
\]

for each \(n \geq 2\). The top horizontal maps are contained in the long exact sequence induced by \(f_\ast: \text{map}(X, X; 1) \to \text{map}(X, Y; f)\) on rational homotopy groups. The bottom horizontal maps are contained in the long exact homology sequence of the map (7). From Lemma 3.1, these sequences are equivalent. 

Remark 3.4. When we refer to the long exact homotopy sequence of a map, we mean this in the sense of [11, Chaps.3,4]: Recall that given a map \(f: X \to Y\), this sequence is as follows:

\[
\cdots \to \pi_n(X) \xrightarrow{f_\#} \pi_n(Y) \to \pi_n(f) \to \pi_{n-1}(X) \to \cdots \to \pi_2(X) \xrightarrow{f_\#} \pi_2(Y).
\]

If \(f\) is the inclusion of a subspace, then the groups \(\pi_n(f)\) are just the usual homotopy groups of a pair. Generally, \(\pi_n(f)\) is defined as homotopy classes of pairs \((g_1, g_2)\) such that the diagram

\[
\begin{array}{ccc}
S^{n-1} & \xrightarrow{g_1} & X \\
\downarrow & & \downarrow f \\
CS^{n-1} & \xrightarrow{g_2} & Y
\end{array}
\]

commutes. Since \(\pi_2(f)\) is not necessarily abelian and we are interested in rationalizing this sequence, we stop at \(\pi_2(Y)\). On the other hand, one can convert \(f\) into a fibration and use the corresponding long exact sequence in homotopy. Either approach suits our purposes and indeed the same sequence of homotopy groups results from either. From the above, we see that if \(\mathcal{F}\) denotes the homotopy fibre of the map \(f_\ast: \text{map}(X, X; 1) \to \text{map}(X, Y; f)\), then for \(n \geq 2\) we have

\[
\pi_{n+1}(f_\ast) \otimes \mathbb{Q} \cong \pi_n(\mathcal{F}) \otimes \mathbb{Q} \cong H_{n+1}(\text{Rel}(\mathcal{M}_f^*)),
\]

where \(\text{Rel}_\ast(\mathcal{M}_f^*)\) is the relative chain complex of the map (7).

The preceding result specializes to give a description of the long exact sequence induced by a general map on rational homotopy groups. The minimal model \(\mathcal{M}_f: \mathcal{M}_Y \to \mathcal{M}_X\) of the map \(f: X \to Y\) also induces a map of chain complexes (8)

\[
(\mathcal{M}_f)^*: \text{Der}(\mathcal{M}_X, \mathbb{Q}, \varepsilon) \to \text{Der}(\mathcal{M}_Y, \mathbb{Q}, \varepsilon).
\]

Theorem 3.5. The long exact sequence induced by \(f: X \to Y\) on rational homotopy groups is equivalent to the long exact homology sequence of the map (8). Specifically, this is a long exact sequence

\[
\begin{array}{ccc}
\cdots & \xrightarrow{H(J)} & H_n(\text{Rel}(\mathcal{M}_f^*)) \\
\downarrow & & \downarrow (\mathcal{M}_f^*) \nu \\
H_n(\text{Der}(\mathcal{M}_X, \mathbb{Q}, \varepsilon)) & \xrightarrow{H((\mathcal{M}_f)^* \nu)} & H_n(\text{Der}(\mathcal{M}_Y, \mathbb{Q}, \varepsilon)) \xrightarrow{H(J)} \cdots
\end{array}
\]
that we terminate in \( H_2(\text{Der}(\mathcal{M}_Y, \mathbb{Q}; \varepsilon)) \), in which \( \text{Rel}_*((\mathcal{M}_f)^*) \) is the relative chain complex of the map (8).

Proof. Argue exactly as in the proof of Theorem 3.3. \( \square \)

Remark 3.6. There is already a standard way to describe the long exact sequence induced by a map on rational homotopy groups, using minimal models. This uses the notion of a so-called K-S model of the map \( \mathcal{M}_f: \mathcal{M}_Y \to \mathcal{M}_X \) [3, Sec.15(d)]. The description we give above, however, is better suited to our purposes. Note that if \( F \) denotes the homotopy fibre of the map \( f: X \to Y \), then for \( n \geq 2 \) we have

\[
\pi_{n+1}(f) \otimes \mathbb{Q} \cong \pi_n(F) \otimes \mathbb{Q} \cong H_{n+1}(\text{Rel}((\mathcal{M}_f)^*)),
\]

where \( \text{Rel}_*((\mathcal{M}_f)^*) \) is the relative chain complex of the map (8). It is perhaps interesting to compare the description given in Theorem 3.5 to the standard description of the long exact sequence in rational homotopy groups of a fibration.

We now identify the \( G \)-sequence within our current framework. Suppose given a DG algebra map \( \phi: A \to B \). Starting from this map, we can construct the following commutative square of DG vector spaces:

\[
\begin{array}{ccc}
\text{Der}_* (B, B; 1) & \xrightarrow{\phi^*} & \text{Der}_* (A, B; \phi) \\
\varepsilon_* & & \varepsilon_* \\
\text{Der}_* (B, Q; \varepsilon) & \xrightarrow{\phi^*} & \text{Der}_* (A, Q; \varepsilon).
\end{array}
\]

In this diagram, \( \varepsilon \) denotes the augmentation of either \( A \) or \( B \), and we have used a decoration to distinguish the lower horizontal map from the upper. On passing to homology and using the naturality of the relative chain complex construction, we obtain the following homology ladder:

\[
\begin{array}{cccc}
\cdots & \xrightarrow{H(J)} & H_{n+1}(\text{Rel}(\phi^*)) & \xrightarrow{H(P)} & H_0(\text{Der}(B, B; 1)) & \xrightarrow{H(\phi^*)} & H_0(\text{Der}(A, B; \phi)) & \cdots \\
& \xrightarrow{H(\varepsilon_*, \varepsilon_*)} & H_0(\text{Der}(B, Q; \varepsilon)) & \xrightarrow{H(\phi^*)} & H_0(\text{Der}(A, Q; \varepsilon))
\end{array}
\]

for \( n \geq 2 \). We supplement Definition 2.7 with the following vocabulary.

Definition 3.7. Suppose \( \phi: A \to B \) is a map of DG algebras. For \( n \geq 3 \) we define the \( n \)th relative evaluation subgroup of \( \phi \) as the image of the map

\[
H(\varepsilon_*, \varepsilon_*): H_n(\text{Rel}(\phi^*)) \to H_n(\text{Rel}(\phi^*)).
\]

We denote it by \( G_n^{rel}(A, B; \phi) \). Then the image of the upper long exact sequence in the lower, of the ladder above, gives a (not necessarily exact) sequence

\[
\cdots \xrightarrow{H(J)} G_{n+1}^{rel}(A, B; \phi) \xrightarrow{H(\hat{\phi})} G_{n+1}(B) \xrightarrow{H(\hat{\phi}^*)} G_n(A, B; \phi) \xrightarrow{H(J)} \cdots
\]

that we terminate in \( G_2(A, B; \phi) \). We refer to this sequence as the \( G \)-sequence of the map \( \phi: A \to B \).

All of the above can be applied to the minimal model \( \mathcal{M}_f: \mathcal{M}_Y \to \mathcal{M}_X \) of the map \( f: X \to Y \). By doing so, and then collecting together previous results, we obtain the following result.
Theorem 3.8. The rationalization of the G-sequence of the map \( f : X \to Y \) (cf. Corollary 2.4), as far as the term \( G_2(Y, X; f) \), is equivalent to the G-sequence of its Sullivan minimal model \( M_f : M_Y \to M_X \), as defined in Definition 3.7.

Proof. Starting from the cube displayed in the proof of Theorem 2.1, we extend each of the four left-to-right maps into their respective long exact sequences. This is then completed into an equivalence of ladders, by defining isomorphisms \( \gamma_n \) and \( \gamma_n' \) to give a commutative square

\[
\begin{array}{ccc}
\pi_n(f) & \otimes & Q \\
\downarrow & & \downarrow \gamma_n \\
\pi_n(f) & \otimes & Q \\
\end{array}
\]

for each \( n \geq 3 \). If these isomorphisms are each defined separately as in Lemma 3.1, using the top and bottom faces of the cube, then a technical problem arises due to the non-natural choice of splittings made there. However, this problem may be surmounted by Lemma 3.9 below.

The result now follows, since the equivalence of ladders restricts to give an equivalence of the corresponding sequences of images. \( \square \)

Lemma 3.9. Suppose given diagrams of vector spaces

\[
\begin{array}{cccc}
X_{n+1} & \to & Y_{n+1} & \to & Z_{n+1} & \to & X_n & \to & Y_n \\
\alpha_{n+1} & \sim & \beta_{n+1} & \sim & \gamma_{n+1} & \sim & \alpha_n & \sim & \beta_n \\
A_{n+1} & \to & B_{n+1} & \to & C_{n+1} & \to & A_n & \to & B_n \\
f_{n+1} & \downarrow & t_{n+1} & \downarrow & u_{n+1} & \downarrow & f_n & \downarrow & v_n \\
\beta_{n+1} & \sim & \gamma_{n+1} & \sim & \alpha_n & \sim & \beta_n \\
A'_n & \to & B'_n & \to & C'_n & \to & A'_n & \to & B'_n \\
v_{n+1} & \downarrow & j_{n+1} & \downarrow & k_{n+1} & \downarrow & v_n' & \downarrow & w_n \\
\alpha'_{n+1} & \sim & \beta_{n+1} & \sim & \gamma_{n+1} & \sim & \alpha_n' & \sim & \beta_n' \\
\end{array}
\]

for \( n \geq 2 \). Suppose both top and bottom faces satisfy all hypotheses of Lemma 3.1, that front and back ladders are commutative, and that the given internal faces commute, that is, \( t_n \circ \alpha_n = \alpha_n' \circ f_n \) and \( u_n \circ \beta_n = \beta_n' \circ g_n \). Then isomorphisms \( \gamma_{n+1} \) and \( \gamma_{n+1}' \) exist, for \( n \geq 2 \), that make the entire diagram commute.

Proof. We proceed as in the proof of Lemma 3.1. However, we must be careful to ensure that the identity \( \psi_{n+1} \circ \gamma_{n+1} = \gamma_{n+1} \circ h_{n+1} \) holds. First, make any choice of splittings \( s_n \) and \( r_n \) of \( k_{n+1} : C_{n+1} \to \text{im}(k_{n+1}) \) and \( r_{n+1} : Z_{n+1} \to \text{im}(r_{n+1}) \) respectively, then define \( \gamma_{n+1} : C_{n+1} \to Z_{n+1} \) as in the proof of Lemma 3.1. As there, \( \gamma_{n+1} \) is an isomorphism that makes the top face of \( (9) \) commute, and also satisfies \( \gamma_{n+1} \circ s_n \circ k_{n+1} = \sigma_n \circ \alpha_n \circ k_{n+1} : C_{n+1} \to Z_{n+1} \). This choice will influence the corresponding choices for the bottom part of diagram \( (9) \). For the splitting \( s_n' \),
consider the following diagram.

\[
\begin{array}{c}
C_{n+1} \xrightarrow{k_{n+1}} \text{im}(k_{n+1}) \xrightarrow{f_n} 0 \\
\downarrow h_{n+1} \quad \downarrow f_n \\
C'_{n+1} \xrightarrow{k'_{n+1}} \text{im}'(k_{n+1}) \xrightarrow{f'_n} 0 \\
\end{array}
\]

Write \( \text{im}(k'_{n+1}) = f_n(\text{im}(k_{n+1})) \oplus K_n \) for a complement \( K_n \). Then because \( k'_{n+1} \circ h_{n+1} \circ s_n = f_n \circ k_{n+1} \circ s_n = f_n : \text{im}(k_{n+1}) \to \text{im}(k'_{n+1}) \), we may choose a splitting \( s'_n : f_n(\text{im}(k_{n+1})) \to \text{im}(h_{n+1} \circ s_n) \) of the restriction of \( k'_{n+1} \) to \( k'_{n+1} : \text{im}(h_{n+1} \circ s_n) \to f_n(\text{im}(k_{n+1})) \) so that \( s'_n \circ f_n \circ k_{n+1} = h_{n+1} \circ s_n \circ k_{n+1} : C_{n+1} \to C'_{n+1} \). Now choose an extension of it over \( K_n \) to a splitting \( s'_n : \text{im}(k'_{n+1}) \to C'_{n+1} \). By choice, we have \( k'_{n+1} \circ s'_n = 1 : \text{im}(k_{n+1}) \to \text{im}(k'_{n+1}) \) and the previous identity makes our choice of splitting “sufficiently natural” for our purposes. A similar argument on the corresponding back face of (9), starting from the chosen splitting \( \sigma_n \), obtains a splitting \( \sigma'_n : \text{im}(r'_{n+1}) \to Z'_{n+1} \) that satisfies \( r'_{n+1} \circ \sigma'_n = 1 : \text{im}(r'_{n+1}) \to \text{im}(r'_{n+1}) \) and \( \sigma'_n \circ t_n \circ r_{n+1} = v_{n+1} \circ \sigma \circ r_{n+1} : X_n \to X'_n \). With these choices of splittings, define \( \gamma'_{n+1} : C'_{n+1} \to Z'_{n+1} \) using \( s'_n \) and \( \sigma'_n \), exactly as in Lemma 3.1. As in that lemma, \( \gamma'_{n+1} \) is an isomorphism that makes the bottom face of (9) commute. It remains to check that \( v_{n+1} \circ \gamma_{n+1} = \gamma'_{n+1} \circ h_{n+1} \) holds. Recall that we have a decomposition \( C_{n+1} = \text{im}(j_{n+1}) \oplus s_n(\text{im}(k_{n+1})) \). On \( \text{im}(j_{n+1}) \), an easy diagram chase in the second cube of (9) gives \( v_{n+1} \circ \gamma_{n+1} \circ j_{n+1} = \gamma'_{n+1} \circ h_{n+1} \circ j_{n+1} \). On \( s_n(\text{im}(k_{n+1})) \), we have the identities

\[
\begin{align*}
v_{n+1} \circ \gamma_{n+1} \circ s_n \circ k_{n+1} &= v_{n+1} \circ \sigma_n \circ \alpha_n \circ k_{n+1} = v_{n+1} \circ \sigma_n \circ r_{n+1} \circ \gamma_{n+1} \\
&= \sigma'_n \circ t_n \circ r_{n+1} \circ \gamma_{n+1} = \sigma'_n \circ t_n \circ \alpha_n \circ k_{n+1} \\
&= \sigma'_n \circ \alpha'_n \circ f_n \circ k_{n+1} = \gamma'_{n+1} \circ s'_n \circ f_n \circ k_{n+1} \\
&= \gamma'_{n+1} \circ h_{n+1} \circ s_n \circ k_{n+1}.
\end{align*}
\]

Notice that we have used all the given commutativity of (9), including that of the internal faces.

In particular, from Theorem 3.8, we obtain the companion result to Corollary 2.5 and Corollary 2.6.

**Corollary 3.10.** Let \( f : X \to Y \) be a map between simply connected complexes of finite type with \( X \) finite. The rationalized \( n \)th relative evaluation subgroup \( G_n^{\text{rel}}(Y_0, X_0; f_0) \cong G_n^{\text{rel}}(Y, X; f) \otimes \mathbb{Q} \) of the map \( f \) is isomorphic to the image of the induced homomorphism

\[
H(\varepsilon_*, \varepsilon_*): H_n(\text{Rel}((Mf)^*)) \to H_n(\text{Rel}((\widehat{Mf})))
\]

for \( n \geq 3 \).

**Remark 3.11.** We comment on the low-end terms in the \( G \)-sequence. In Theorem 3.5 and Theorem 3.3 we terminate our long exact sequences at the terms corresponding to \( \pi_2(\text{map}(X, Y; f)) \) and \( \pi_2(Y) \) respectively. This is because we need a simply connected hypothesis to ensure our combination of rationalization and minimal model techniques remains valid. As a result, our algebraic description of the rationalized
$G$-sequence terminates at the term corresponding to $G_2(X, Y; f)$. Now in Theorem 3.5, we require $X$ to be simply connected and finite. By [2, Th.III], this implies $G_{2i}(X) \otimes \mathbb{Q} = 0$ for each $i$. Therefore, under our hypotheses, the rationalized $G$-sequence of a map $f: X \to Y$ should be considered as 5-term (not necessarily exact) sequences

$$
\begin{array}{cccccc}
0 & \longrightarrow & G_{2n}(Y, X; f) \otimes \mathbb{Q} & \longrightarrow & G_{2n}^{rel}(Y, X; f) \otimes \mathbb{Q} & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
G_{2n-1}(X) \otimes \mathbb{Q} & \longrightarrow & G_{2n-1}(Y, X; f) \otimes \mathbb{Q} & \longrightarrow & G_{2n-1}^{rel}(Y, X; f) \otimes \mathbb{Q} & \longrightarrow 0
\end{array}
$$

for $n \geq 2$. Our algebraic description of the rationalized $G$-sequence given by Theorem 3.8 includes all these 5-term sequences. The “sporadic” low-end term $G_2(Y, X; f) \otimes \mathbb{Q}$ is best computed by using Corollary 2.6.

Before turning to some applications of our algebraic description of the rationalized $G$-sequence, we give one more description of a long exact homotopy sequence. Let $\text{map}_*(X, Y; f)$ denote the component of $f$ in the space of basepoint-preserving functions from $X$ to $Y$. Then we have the evaluation fibration sequence

$$
\text{map}_*(X, Y; f) \longrightarrow \text{map}(X, Y; f) \longrightarrow \omega Y.
$$

We will describe the long exact sequence on rational homotopy groups induced by this fibration.

Let $\phi: A \to B$ be a DG algebra map. Observe that the augmentation $\varepsilon: A \to \mathbb{Q}$ for $A$ induces a surjection $\varepsilon_*: \text{Der}_*(A, B; \phi) \to \text{Der}_*(A, \mathbb{Q}; \varepsilon)$. Let

$$
\widetilde{\text{Der}}_* (A, B; \phi) = \ker (\varepsilon_*: \text{Der}_*(A, B; \phi) \to \text{Der}_*(A, \mathbb{Q}; \varepsilon))
$$

and let $I: \widetilde{\text{Der}}_* (A, B; \phi) \to \text{Der}_*(A, B; \phi)$ denote the inclusion. The resulting short exact sequence of DG vector spaces

$$
0 \longrightarrow \widetilde{\text{Der}}_* (A, B; \phi) \xrightarrow{I} \text{Der}_*(A, B; \phi) \xrightarrow{\varepsilon_*} \text{Der}_*(A, \mathbb{Q}; \varepsilon) \longrightarrow 0.
$$

gives a long exact sequence on homology, in the usual way, of the form

$$
\begin{array}{cccccc}
\cdots & \xrightarrow{\Delta_*} & H_n(\widetilde{\text{Der}}(A, B; \phi)) & \xrightarrow{H(I)} & H_n(\text{Der}(A, B; \phi)) & \xrightarrow{H(\varepsilon_*)} & H_n(\text{Der}(A, \mathbb{Q}; \varepsilon)) & \cdots
\end{array}
$$

for $n \geq 2$. Call this sequence the long exact derivation homology sequence of the DG algebra map $\phi: A \to B$.

**Theorem 3.12.** The long exact sequence induced by the evaluation fibration (10) on rational homotopy groups is equivalent to the long exact derivation homology sequence (11) of the map $M_f: M_Y \to M_X$. Specifically, this is a long exact sequence

$$
\begin{array}{cccccc}
\cdots & \xrightarrow{H(\varepsilon_*)} & H_{n+1}(\text{Der}(M_Y, \mathbb{Q}; \varepsilon)) & \xrightarrow{\Delta_*} & H_n(\text{Der}(M_Y, M_X; M_f)) & \xrightarrow{H(I)} & H_n(\text{Der}(M_Y, M_X; M_f)) & \xrightarrow{H(\varepsilon_*)} & \cdots
\end{array}
$$

that terminates at $H_2(\text{Der}(M_Y, \mathbb{Q}; \varepsilon))$. 
Proof. Adapt the proof of Theorem 3.3, starting here with the right face of the cube displayed in the proof of Theorem 2.1. \( \square \)

**Corollary 3.13.** Let \( X \) and \( Y \) be simply connected spaces with \( X \) finite. Then the rational homotopy groups of the based function space \( map_{\ast}(X,Y;f) \) are given by

\[
\pi_n(map_{\ast}(X,Y;f)) \otimes \mathbb{Q} \cong H_n(\text{Der}(M_Y,M_X;M_f))
\]

for \( n \geq 2 \).

**Remark 3.14.** We cannot claim naturality in the equivalence of Theorem 3.12, or the other equivalences and isomorphisms of this section, due to the choices involved in Lemmas 3.1 and 3.9. We believe it should be possible to establish natural equivalences, by expliciting defining the relevant isomorphisms as we did in the proof of Theorem 2.1. However, we have chosen to argue at the expense of naturality, to spare extensive details beyond those already involved in the proof of Theorem 2.1.

4. Examples, Computations, and Further Consequences

We illustrate the effectiveness of the framework established in the previous two sections with examples. First, we give a composite example that includes specific computation of many of the ingredients of the above. Our example is one in which the \( G \)-sequence of a map fails to be exact (after rationalization) at each of the three types of term that occur.

We begin with some notational conventions. Suppose that \((A,d_A)\) and \((B,d_B)\) are minimal algebras, with \( A = \Lambda(W) \) and \( B = \Lambda(V) \) for suitable graded vector spaces \( W \) and \( V \). Let \( \phi: A \to B \) be a fixed DG algebra map. Since any linear map \( W \to B \) extends in a unique way to a \( \phi \)-derivation, and vice-versa we may restrict a \( \phi \)-derivation to \( W \), we may identify the space \( \text{Hom}_*(W,B) \) of negative degree linear maps with \( \text{Der}_*(A,B;\phi) \) (in a way that depends on \( \phi \)). This point of view, whereby a derivation is specified on generators and then extended to the whole algebra, is one that we will invariably adopt in any practical calculation.

Now suppose given a basis \( \{w_1, w_2, w_3, \ldots \} \) for \( W \) and an element \( P \in B \) with \( |P| < |w_i| \) we will write \( P\partial w_i \) for the \( \phi \)-derivation carrying \( w_i \) to \( P \) and vanishing on the other \( w_j \). Thus any derivation can be expressed as a sum \( \sum_i P_i\partial w_i \). When \( B = \mathbb{Q} \), we write \( w_j^* \) rather than \( 1 \partial w_j \) for the derivation dual to \( w_j \).

**Example 4.1.** Let \( f = (f_1,f_2): \mathbb{H}P^2 \to S^8 \times \mathbb{H}P^4 \) be the map with coordinate functions \( f_1: \mathbb{H}P^2 \to S^8 \) obtained by pinching out the bottom cell and \( f_2: \mathbb{H}P^2 \to \mathbb{H}P^4 \) the inclusion. We will use our framework to compute various terms from the long exact sequences corresponding to Theorem 3.3 and Theorem 3.5. Denote \( \mathbb{H}P^2 \) by \( X \) and \( S^8 \times \mathbb{H}P^4 \) by \( Y \), thus \( f: X \to Y \). Our computation will show, using Theorem 3.8, that the \( G \)-sequence of \( f \) is non-exact at the terms \( G_4(Y,X;\phi), G_8^{rel}(Y,X;\phi), \) and \( G_{11}(X) \).

First, \( M_X = \Lambda(x_4,x_{11}) \), with differential given on generators by \( d(x_4) = 0 \), and \( d(x_{11}) = x_4^3 \), and \( M_Y = \Lambda(y_8,y_{15},y_{19}) \) with differential \( d(y_8) = 0 \), \( d(y_{15}) = y_8^2 \), \( d(y_{19}) = 0 \), and \( d(y_{15}) = y_8^2 \). In both models, subscripts denote degrees. Then the Sullivan model of \( f \), which we denote by \( \phi: M_Y \to M_X \), is given on generators by \( \phi(y_8) = x_4^3, \phi(y_{15}) = x_4^2x_{11}, \phi(y_{19}) = 0, \phi(y_{19}) = x_4^3x_{11} \).

For degree reasons, \( \text{Der}_i(M_X,M_X;1) = 0 \) unless \( i = 3,4,7 \) or 11. Furthermore, \( \text{Der}_*(M_X,M_X;1) \) is spanned by the derivations \( x_4^2\partial x_{11}, x_4^2, x_4\partial x_{11}, \) and \( x_{11}^2 \) of
number of standard methods. To this point, our observations are both well-known, and also easily obtained by a further, it is evident that $G$ that $\pi_i(\text{map}(X,X;1)) \otimes \mathbb{Q} \cong H_i(\text{Der}(\mathcal{M}_Y,\mathcal{M}_X;1)) = \begin{cases} \mathbb{Q} & \text{if } i = 7, 11 \\ 0 & \text{otherwise} \end{cases}$

with the non-zero cohomology in degrees 4 and 11 generated by cocycles $x_4^*$ and $x_{11}^*$, respectively. Given these generators, we see that $H(\varepsilon_\ast): H_i(\text{Der}(\mathcal{M}_Y,\mathcal{M}_X;1)) \to H_i(\text{Der}(\mathcal{M}_X,\mathbb{Q};\varepsilon))$, that is, the homomorphism $\omega_\# \otimes \mathbb{Q}: \pi_i(\text{map}(X,X;1)) \otimes \mathbb{Q} \to \pi_i(X) \otimes \mathbb{Q}$ induced by the evaluation map on rational homotopy groups, is an isomorphism in degree 11 and is zero in all other degrees. It follows from Theorem 2.1—see Corollary 2.5—that $G_{11}(\mathcal{M}_X) = 0$ other than in degree 11, where we have $G_{11}(\mathcal{M}_X) \cong \mathbb{Q}$. Up to this point, our observations are both well-known, and also easily obtained by a number of standard methods.

We now show that the rationalized $G$-sequence is non-exact at the $G_{11}(\mathcal{M}_X)$ term. Recall that this term of the $G$-sequence, together with its adjacent terms, is obtained from the diagram

$$
\xymatrix{ \text{Rel}_{12}(\phi^\ast) \ar[r]^P & \text{Der}_{11}(\mathcal{M}_X,\mathcal{M}_X;1) \ar[r]^-{\phi^\ast} & \text{Der}_{11}(\mathcal{M}_Y,\mathcal{M}_X;\phi) \\
\text{Rel}_{12}(\tilde{\phi}^\ast) \ar[r]^\tilde{P} & \text{Der}_{11}(\mathcal{M}_X,\mathbb{Q};\varepsilon) \ar[r]^-{\tilde{\phi}^\ast} & \text{Der}_{11}(\mathcal{M}_Y,\mathbb{Q};\varepsilon),}
$$

by passing to homology and then considering the image of the top sequence in the bottom. A brute force calculation will display the result, but we opt to argue at a more general level so as to indicate some reason for non-exactness. It is evident that $H(\tilde{\phi}^\ast) \circ H(\varepsilon_\ast)(\langle x_{11}^1 \rangle) = 0 \in H_{11}(\text{Der}(\mathcal{M}_Y,\mathbb{Q};\varepsilon))$—indeed, this latter term is zero, since it is isomorphic to $\pi_{11}(Y) \otimes \mathbb{Q} = 0$. The key point for non-exactness here, however, is that in the top sequence we have $H(\phi^\ast)(\langle x_{11}^1 \rangle) \neq 0 \in H_{11}(\text{Der}(\mathcal{M}_Y,\mathcal{M}_X;\phi))$. In fact, a straightforward check shows that $\phi^\ast(x_{11}^1) = x_3 \partial y_{15} + x_2 \partial y_{19}$. Since $\text{Der}_{12}(\mathcal{M}_Y,\mathcal{M}_X;\phi) = 0$, there are no non-zero boundaries in degree 11 and hence $H(\phi^\ast)(\langle x_{11}^1 \rangle) \neq 0$. Consequently, $\langle x_{11}^1 \rangle$ cannot be in the image of $H(P)$ in the top sequence. Therefore, since $H(\varepsilon_\ast)$ is an isomorphism in degree 11, $H(\varepsilon_\ast)(\langle x_{11}^1 \rangle) = \langle x_{11}^1 \rangle$ cannot be in the image of $H(\tilde{P}) \circ H(\varepsilon_\ast,\varepsilon_\ast)$. It follows from these facts that $\langle x_{11}^1 \rangle \in \text{ker}(\tilde{P})$ is a non-zero element in the kernel of $H(\tilde{\phi}^\ast)$ and yet is not in the image of $H(\tilde{P}): G_{12}^{11}(\mathcal{M}_Y,\mathcal{M}_X;\phi) \to G_{11}(\mathcal{M}_X)$. 

degree 3, 4, 7, and 11 respectively. An easy computation reveals that $D(x_4^1) = -3x_4^1 \partial x_{11}$, but that $x_4 \partial x_{11}$ and $x_{11}$ are both (non-exact) cocycles. It follows from Theorem 2.1 that

$$
\pi_i(\text{map}(X,X;1)) \otimes \mathbb{Q} \cong H_i(\text{Der}(\mathcal{M}_X,\mathcal{M}_X;1)) = \begin{cases} \mathbb{Q} & \text{if } i = 7, 11 \\ 0 & \text{otherwise} \end{cases}
$$

Next consider the term $G_1(M_Y, M_X; \phi)$: Before passing to homology, the relevant diagram is the following:

$$
\begin{array}{cccc}
\text{Der}_4(M_X, M_X; 1) & \xrightarrow{\phi^*} & \text{Der}_4(M_Y, M_X; \phi) & \xrightarrow{J} & \text{Rel}_4(\phi^*) \\
\downarrow{\varepsilon_*} & & \downarrow{\varepsilon_*} & & \downarrow{(\varepsilon_*, \varepsilon_*)} \\
\text{Der}_4(M_X, Q; \varepsilon) & \xrightarrow{\tilde{\phi}^*} & \text{Der}_4(M_Y, Q; \varepsilon) & \xrightarrow{J} & \text{Rel}_4(\tilde{\phi}^*)
\end{array}
$$

The derivation $\theta = y_1^* + 5x_4x_11\partial y_{19} \in \text{Der}_4(M_Y, M_X; \phi)$ is a cocycle, as is easily checked. Under $H(\varepsilon_*): H_4(\text{Der}(M_Y, M_X; \phi)) \to H_4(\text{Der}(M_Y, Q; \varepsilon))$, we have $H(\varepsilon_*)(\langle \theta \rangle) = \langle y_1^* \rangle \neq 0$. Since $\langle y_1^* \rangle = H(\tilde{\phi}^*)(\langle x_1^* \rangle)$, it follows that $H(\tilde{J})(\langle y_1^* \rangle) = 0$. As we noted above, however, $G_4(M_X) = 0$. Therefore, $\langle y_1^* \rangle \in G_4(M_Y, M_X; \phi)$ is a non-zero element in the kernel of $H(\tilde{J}): G_4(M_Y, M_X; \phi) \to G_3^c(M_Y, M_X; \phi)$ that is not in the image of $H(\tilde{\phi}^*)$: $G_4(M_Y) \to G_4(M_Y, M_X; \phi)$.

Finally, consider the term $G^c_3(M_Y, M_X; \phi)$: Here, we begin with the following diagram:

$$
\begin{array}{ccc}
\text{Der}_8(M_Y, M_X; \phi) & \xrightarrow{J} & \text{Rel}_8(\phi^*) & \xrightarrow{P} & \text{Der}_7(M_X, M_X; 1) \\
\downarrow{\varepsilon_*} & & \downarrow{\varepsilon_*} & & \downarrow{\varepsilon_*} \\
\text{Der}_8(M_Y, Q; \varepsilon) & \xrightarrow{J} & \text{Rel}_8(\tilde{\phi}^*) & \xrightarrow{\tilde{P}} & \text{Der}_7(M_X, Q; \varepsilon)
\end{array}
$$

We find that $(2x_4\partial x_{11}, y_8^* - 2x_{11}\partial y_{19}) \in \text{Rel}_8(\phi^*)$ is a cocycle that has a non-zero image in $H(\varepsilon_*, \varepsilon_*): H_8(\text{Rel}(\phi^*)) \to H_8(\text{Rel}(\tilde{\phi}^*))$. Furthermore, it is evident that $H(\tilde{J}) \circ \text{Rel}_8(\phi^*)(2x_4\partial x_{11}, y_8^* - 2x_{11}\partial y_{19}) = H(\tilde{J})(\langle 0, y_8^* \rangle) = 0$—indeed, $H_7(\text{Der}(M_X, Q; \varepsilon)) \cong \pi_7(X) \otimes \mathbb{Q} = 0$. To see that $\langle 0, y_8^* \rangle$ is not in the image of $H(\tilde{J}): G_8(M_Y, M_X; \phi) \to G_7^c(M_Y, M_X; \phi)$, we will compute $G_8(M_Y, M_X; \phi)$ to be zero.

A general derivation $\gamma \in \text{Der}_8(M_Y, M_X; \phi)$ can be written as

$$\gamma = \lambda_1 y_8^* + \lambda_2 x_{11}\partial y_{19},$$

for coefficients $\lambda_i \in \mathbb{Q}$. To find the cocycles of this form, we first observe that $D(\gamma)$ must vanish on the generators $y_8$ and $y_4$, for degree reasons. On $y_{15}$, we compute as follows:

$$D(\gamma)(y_{15}) = (d_X(\gamma) - (\gamma)d_Y)(y_{15})$$

$$= 0 - \gamma(y_8^*)$$

$$= -2\gamma(y_8)\phi(y_8)$$

$$= -2\lambda_1 x_{11}^2 y_8^*.$$ 

Thus, if $\gamma$ is a cocycle, then we must have $\lambda_1 = 0$. A similar computation shows that $D(\gamma)(y_{19}) = \lambda_2 x_{11}^3 y_8^*$, and thus that there are no non-zero cocycles in $\text{Der}_8(M_Y, M_X; \phi)$. In summary, we have computed that

$$\pi_8(\text{map}(X, Y; f)) \otimes \mathbb{Q} \cong H_8(\text{Der}(M_Y, M_X; \phi)) = 0.$$

This last part of our computation is easily confirmed using the result of Corollary 2.11. It follows, of course, that $G_8(M_Y, M_X; \phi) = 0$ and, in particular, that $\langle (y_8^*, 0) \rangle$ is not in the image of $H(\tilde{J}): G_8(M_Y, M_X; \phi) \to G_7^c(M_Y, M_X; \phi)$. 


Remarks 4.2. The first example of a non-exact $G$-sequence, given in [24], was in dimension one. A higher dimensional example was produced later in [25]. With the approach illustrated in the above example, it is straightforward to produce higher dimensional examples of non-exact rationalized $G$-sequences. Observe that non-exact ordinary $G$-sequences are produced as a result, since if a sequence of abelian groups is not exact after tensoring with $\mathbb{Q}$ then it was not exact to begin with. Thus, non-exactness rationally implies non-exactness integrally.

Although the $G$-sequence in general is non-exact, there are certain situations in which it is perfectly well behaved, at least after rationalization. We now mention one such situation. We say that a space $X$ is an $H_0$-space if its rational cohomology algebra is a free graded algebra (exterior algebra on the odd-degree generators tensored with a polynomial algebra on the even-degree generators). Equivalently, we could require that $X$ be an $H$-space after rationalization, whence such a space is also referred to as a rational $H$-space. Recall also the definition of an $F_0$-space from before Corollary 2.11.

Theorem 4.3. Let $f : X \to Y$ be any map from an $F_0$-space $X$ to an $H_0$-space $Y$ that induces the zero homomorphism on rational homotopy groups. Then the rationalized $G$-sequence splits into short exact sequences

$$0 \longrightarrow G_{n+1}(Y, X; f) \otimes \mathbb{Q} \overset{J}{\longrightarrow} G^r_{n+1}(Y, X; f) \otimes \mathbb{Q} \overset{P}{\longrightarrow} G_n(X) \otimes \mathbb{Q} \longrightarrow 0$$

for each $n \geq 2$.

Proof. Our assumption that $f_\# \otimes \mathbb{Q} = 0$ means that the long exact sequence induced by $f$ on rational homotopy groups splits. Furthermore, since $Y$ is an $H_0$-space, we have $G_n(Y) \otimes \mathbb{Q} = \pi_n(Y) \otimes \mathbb{Q}$ for each $n$. It follows that $G_n(Y, X; f) \otimes \mathbb{Q} = \pi_n(Y) \otimes \mathbb{Q}$ for each $n$. From these general considerations, we have short exact sequences

$$0 \longrightarrow G_{n+1}(Y, X; f) \otimes \mathbb{Q} \overset{J}{\longrightarrow} \pi_{n+1}(f) \otimes \mathbb{Q} \overset{P}{\longrightarrow} \pi_n(X) \otimes \mathbb{Q} \longrightarrow 0$$

for $n \geq 2$. To sharpen this to the statement of the theorem, we work within our minimal model framework.

We use some results of Halperin [9], on the rational homotopy of an $F_0$-space. These state that $X$ has minimal model $(\mathcal{M}_X, d_X)$ of the form $\mathcal{M}_X = \Lambda(x_1, \ldots, x_n) \otimes \Lambda(y_1, \ldots, y_m)$ where $|x_i|$ is even, $|y_j|$ is odd, $d_X(x_i) = 0$ and $d_X(y_j) \in \Lambda(x_1, \ldots, x_n)$. Furthermore, the cohomology is evenly graded, and any cocycle in $I(y_1, \ldots, y_n)$, the ideal of $\mathcal{M}_X$ generated by the $y_i$, is exact. It is well-known that the minimal model of an $H_0$-space takes the form $\mathcal{M}_Y = \Lambda(z_1, z_2, \ldots)$, with trivial differential. The map $f : X \to Y$ has Sullivan minimal model $\phi : \mathcal{M}_Y \to \mathcal{M}_X$ that is determined by the $\phi(z_i)$. Since each $z_i$ is a cocycle, it follows that each $\phi(z_i) \in \mathcal{M}_X$ is a cocycle. From the results of Halperin mentioned above, we can write $\phi(z_i) = \chi_i + d(\xi_i)$, for suitable $\chi_i \in \Lambda(x_1, \ldots, x_n)$ and $\xi_i \in I(y_1, \ldots, y_n)$. The assumption that $f$ induces zero on rational homotopy groups translates into the further restriction that each $\chi_i$ is decomposable.

The short exact sequences (12), translated into our derivation setting, correspond to short exact sequences

$$0 \to G_{n+1}(\mathcal{M}_Y, \mathcal{M}_X; \phi) \overset{H(\tilde{\phi})}{\longrightarrow} H_{n+1}(\text{Rel}(\tilde{\phi}^*)) \overset{H(\tilde{\phi})}{\longrightarrow} H_n(\text{Der}(\mathcal{M}_X, \mathbb{Q}; \tilde{\varepsilon})) \to 0$$
We first prove that each map $H(\tilde{P})$ restricts to give a surjection

$$H(\tilde{P}): G_{n+1}^{rel}(\mathcal{M}_Y, \mathcal{M}_X; \phi) \to G_n(\mathcal{M}_X).$$

The Gottlieb elements of $\mathcal{M}_X$ are precisely the $y_j^*$, dual to the odd-degree generators of $\mathcal{M}_X$. This can be seen from the description of the Gottlieb elements given in Corollary 2.5, together with the facts about the minimal model of $X$ recalled earlier.

Now write $\phi(z_i) = \chi_i + d(\xi_i)$ as above and, for each $j$, define a derivation

$$\theta_j = -\sum_i y_j^*(\xi_i) \partial z_i \in \text{Der}_{[y_j]_1+1}(\mathcal{M}_Y, \mathcal{M}_X; \phi).$$

Then $D(\theta_j) = -\sum_i d(y_j^*(\xi_i)) \partial z_i$, since the differential in $\mathcal{M}_Y$ is trivial. On the other hand, we have

$$\phi^*(y_j^*(z_i)) = y_j^*(\phi(z_i)) = y_j^*(\chi_i + d(\xi_i)) = 0 + y_j^*(d(\xi_i)) = -d(y_j^*(\xi_i)),$$

with the last step following because $y_j^*$ is a $D$-cycle. Consequently, $(y_j^*, \theta_j) \in \text{Rel}_{[y_j]_1+1}(\phi^*)$ is a $\delta$-cycle. Since $H(\tilde{P}) \circ H(\varepsilon_*, \varepsilon_*)(\langle y_j^*, \theta_j \rangle) = y_j^*$, it follows that

$$H(\tilde{P})$$

does restrict to the desired surjection.

The map $J$ is injective on rational homotopy groups, as we have already observed, and therefore restricts to an injection in the rationalized $G$-sequence. So it only remains to show exactness at the $G_{n+1}^{rel}(Y, X; f)$ terms. Suppose that

$$H(\varepsilon_*, \varepsilon_*)(\langle \theta, \psi \rangle) \in \ker(H(\tilde{P}): G_{n+1}^{rel}(\mathcal{M}_Y, \mathcal{M}_X; \phi) \to G_n(\mathcal{M}_X)),$$

for some cocycle $(\theta, \psi) \in \text{Rel}_{n+1}(\phi^*)$. The fact that $H(\tilde{P})(\langle \varepsilon_*(\theta), \varepsilon_*(\psi) \rangle) = 0$ implies that $\varepsilon_*(\theta) = 0$. Thus $(\varepsilon_*, \varepsilon_*)(\langle \theta, \psi \rangle) = (0, \varepsilon_*(\psi))$. Now define a derivation $\overline{\psi} \in \text{Der}_{n+1}(\mathcal{M}_Y, \mathcal{M}_X; \phi)$ by setting $\overline{\psi} = \psi$ on generators of $\mathcal{M}_Y$ of degree $n+1$ and $\overline{\psi} = 0$ on all other generators of $\mathcal{M}_Y$. It is easily seen that $\overline{\psi}$ is a cycle. Indeed, $d_x \overline{\psi} = 0$ on all generators of $\mathcal{M}_Y$, since $\overline{\psi}$ has non-zero image only in degree zero, and $\overline{\psi}d_y = 0$ since $d_y = 0$. Thus $H(\varepsilon_*)(\langle \overline{\psi} \rangle) \in G_{n+1}(\mathcal{M}_Y, \mathcal{M}_X; \phi)$ satisfies

$$H(\tilde{J}) \circ H(\varepsilon_*)(\langle \overline{\psi} \rangle) = (0, \varepsilon \circ \overline{\psi}) = (0, \varepsilon \circ \psi) = H(\varepsilon_*, \varepsilon_*)(\langle \theta, \psi \rangle).$$

That is, $\ker(H(\tilde{P})) \cap G_{n+1}^{rel}(\mathcal{M}_Y, \mathcal{M}_X; \phi) \subseteq H(\tilde{J})(G_{n+1}(\mathcal{M}_Y, \mathcal{M}_X; \phi))$ and the rationalized $G$-sequence is exact at each $G_{n+1}^{rel}(Y, X; f)$ term.

**Remark 4.4.** Various conditions are known, under which the $G$-sequence of a map $f: X \to Y$ is exact. For instance, it is exact when $f$ is null-homotopic [14], and when $f$ is a homotopy monomorphism [26]. The hypotheses of Theorem 4.3 are well-suited for rational homotopy theory. Both types of space are well-known, and it is easy to give examples to which the theorem applies. In fact, for fixed $X$ and $Y$, the maps to which it applies are classified up to rational homotopy by the decomposable rational cohomology of $X$ that occurs in those (even) degrees in which $Y$ has a generator of rational cohomology. We emphasize that the $H_0$-space $Y$ must be allowed to have polynomial generators in rational cohomology, and hence be infinite-dimensional, otherwise the theorem reduces to the case in which the map $f$ is rationally null-homotopic. Furthermore, the hypothesis that $f$ be zero on rational homotopy groups is necessary. For example, the map $f: S^4 \to \mathbb{H}P^\infty$, given by inclusion of the bottom cell, has a non-exact rationalized $G$-sequence, as is easily confirmed by computations similar to those of Example 4.1.
Since the $G$-sequence of a map $f : X \to Y$ is a boundary sequence, but not usually an exact sequence, it is natural to consider its homology. This gives the so-called $\omega$-homology of $f$ [14]. In general, one obtains an $\omega$-homology group at each of the three types of term. In the following, we restrict our attention to the $\omega$-homology that occurs at the Gottlieb group term $G_\omega(X)$, denoted $H^n_{\omega}(X;Y;f)$ in [14]. Thus we consider the sub-quotients of the Gottlieb groups $G_\omega(X)$ defined by

$$H^n_{\omega}(X;Y;f) = \ker\{f_\#: G_n(X) \to G_n(Y;X;f)\} / \text{im}\{P_\#: G^n_{n+1}(Y,X;f) \to G^n(X)\}.$$ 

When $Y$ is an $H_0$-space, the rational $\omega$-homology of $f : X \to Y$ is related to the negative derivations on the rational cohomology of $X$ that are induced by derivations on the minimal model. To be precise, define a linear map of degree zero

$$\varphi_X: H_\omega(\text{Der}(M_X,M_X;1)) \to \text{Der}_*(H^*(Y;\mathbb{Q}), H^*(X;\mathbb{Q});1)$$

by the rule $\varphi_X(\langle \theta \rangle)(\langle \chi \rangle) = \langle \theta(\chi) \rangle$, for $\theta$ a cycle in $\text{Der}_*(M_X,M_X;1)$ and $\chi$ a cocycle in $M_X$. It is straightforward to check that $\varphi_X$ is well-defined. (cf. [7, Prop.1.6]. In fact, $\varphi_X$ is a morphism of graded Lie algebras.)

In the next result, and the example that follows it, we illustrate that the rationalized $G$-sequence may be exact at all occurrences of one type of term, while failing to be exact at the other types of term. In other words, the rational $\omega$-homology of a map may be zero at one type of term, yet non-zero at the other types of term.

**Theorem 4.5.** Let $X$ be a finite complex for which the map $\varphi_X$ defined above is trivial and let $Y$ be an $H_0$-space. Then $H^n_{\omega}(X;Y;f) \otimes \mathbb{Q} = 0$ for any map $f : X \to Y$.

**Proof.** Since $Y$ is an $H_0$-space, its minimal model $M_Y \cong H^*(Y;\mathbb{Q})$ has trivial differential. Let $\phi: H^*(Y;\mathbb{Q}) \to M_X$ denote the minimal model of $f$. For a derivation $\theta \in \text{Der}_*(H^*(Y;\mathbb{Q}), M_X;\phi)$, we have $D(\theta) = \pm d_X \theta$. Using this observation, we obtain a map

$$\mu: H_\omega(\text{Der}_*(H^*(Y;\mathbb{Q}), M_X;\phi)) \to \text{Der}_*(H^*(Y;\mathbb{Q}), H^*(X;\mathbb{Q});H(\phi)),$$

defined by $\mu([\theta]) = [\theta(\chi)]$. Using the preceding observation, together with the free-ness of $H^*(Y;\mathbb{Q})$, it is straightforward to check that $\mu$ is an isomorphism. Furthermore, the following diagram commutes:

$$\begin{array}{ccc}
H_\omega(M_X,M_X;1) & \cong & H_\omega(\text{Der}(H^*(Y;\mathbb{Q}), M_X;\phi)) \\
\varphi_X & & \mu \\
\text{Der}_*(H^*(X;\mathbb{Q}), H^*(X;\mathbb{Q});1) & \text{(H(\phi))} & \text{Der}_*(H^*(Y;\mathbb{Q}), H^*(X;\mathbb{Q});H(\phi))
\end{array}$$

Therefore, the assumption that $\varphi_X = 0$ implies that the top map $H(\phi^*)$ in the above diagram is zero. A straightforward diagram chase using the homology ladder that defines the rationalized $G$-sequence now gives the result. 

**Example 4.6.** Following Theorem 4.3 we remarked that the cellular inclusion $S^4 \to \mathbb{H}P^\infty$ does not have an exact rationalized $G$-sequence. However, it does satisfy the hypotheses of Theorem 4.5, since here $X = S^4$ has the property that all derivations of the cohomology algebra are trivial.
Remark 4.7. The hypothesis on $X$ in Theorem 4.5, that $\varphi_X = 0$, deserves some comment. First, we observe that the nature of the hypothesis distinguishes structure at the minimal model level from structure at the cohomology level. This is a distinction that is made in rational homotopy for a wide variety of structures. Next, we observe that this condition is satisfied for many, if not all, $F_0$-spaces $X$. Indeed, Conjecture 2.10—the long-standing conjecture of Halperin concerning $F_0$-spaces—is equivalent to the assertion that all negative-degree derivations on the cohomology algebra of an $F_0$-space are trivial (see [20] for details). Whenever this conjecture is true—and it has been verified in many cases—obviously we have $\varphi_X = 0$. Therefore, Theorem 4.5 can be compared with Theorem 4.3, as a result with weaker hypotheses, and correspondingly weaker conclusion. Finally, we note that the map $\varphi_X$ makes an appearance in a completely different context, in the work of Belegradek and Kapovitch [1].

Our last set of results relate directly to Conjecture 2.10. First, we observe that for an inclusion of a summand of a product, the $G$-sequence behaves in a particularly nice way. Since it is no harder to do so, we state and prove this result in the integral setting.

Proposition 4.8. Suppose that $i_1: X \to X \times B$ is the inclusion into the first summand. Then the $G$-sequence of $i_1$ is exact, and furthermore reduces to split short exact sequences

$$0 \longrightarrow G_n(X) \xrightarrow{(i_1)_#} G_n(X \times B, X; i_2) \xrightarrow{(p_2)_#} \pi_n(B) \longrightarrow 0,$$

where $p_2: X \times B \to B$ is projection onto the second summand and the splitting is induced by inclusion into the second summand $i_2: B \to X \times B$.

Proof. This follows from results in [38] (see also [34]), but we give a brief argument here. First, let $X \to E \to B$ be any fibre sequence. Hilton’s excision homomorphism for relative homotopy groups gives an isomorphism $\pi_*(j) \cong \pi_*(B)$ [11, Chap.3]. Thus we may view $G^*_{rel}(E, X; j)$ as a subgroup of $\pi_*(B)$ and the $G$-sequence of the fibre inclusion $j$ as a subsequence the long exact homotopy sequence of the fibration.

Now apply this remark to the trivial fibration $X \xrightarrow{i_1} X \times B \xrightarrow{p_2} B$. The inclusion $i_2: B \to X \times B$ induces a splitting of the long exact homotopy sequence of this trivial fibration in the usual way. The result now follows from the observation that $(i_2)_#(\pi_*(B)) \subseteq G_n(X \times B, X; p_2)$, as is easily established from the definitions. \qed

Of course, this result and its proof can be rationalized, and it is in the rational setting that we will use it. Conjecture 2.10 concerns fibrations $X \to E \to B$ with fibre an $F_0$-space and arbitrary base. However, it is well-known how to reduce the conjecture to consideration of such fibrations with base an odd-dimensional sphere [20]. Furthermore, in [17] it is pointed out that, for such fibrations with base an odd-dimensional sphere, Halperin’s conjecture actually asserts that the fibration should be trivial.

From these remarks, we see that a necessary condition for Conjecture 2.10 to be true is that any fibration $X \to E \to S^{2r+1}$ with $X$ an $F_0$-space must have a fibre inclusion whose $G$-sequence reduces to the split short exact sequences corresponding
via Proposition 4.8 to the inclusion \( i_1 : X \to X \times S^{2r+1} \). Perhaps surprisingly, the converse is true.

**Theorem 4.9.** Let \( X \xrightarrow{j} E \xrightarrow{\pi} S^{2r+1} \) be any fibration with \( X \) an \( F_0 \)-space. The following are equivalent:

1. The fibration is rationally TNCZ, that is, \( j^* : H^*(E;\mathbb{Q}) \to H^*(X;\mathbb{Q}) \) is surjective.
2. The rationalized \( G \)-sequence of the fibre inclusion reduces to split short exact sequences

\[
0 \to G_n(X) \otimes \mathbb{Q} \xrightarrow{(j)_\#} G_n(E,X;j) \otimes \mathbb{Q} \xrightarrow{(p)_\#} \pi_n(S^{2r+1}) \otimes \mathbb{Q} \to 0
\]

for each \( n \geq 2 \).

**Proof.** The implication (1) \( \Rightarrow \) (2) follows by the remarks preceding the enunciation. We prove (2) \( \Rightarrow \) (1). Suppose the fibration \( X \to E \to S^{2r+1} \) has minimal model

\[
\Lambda(u) \xrightarrow{i} \Lambda(u) \otimes \Lambda V, D \xrightarrow{\pi} (\Lambda V, d),
\]

where \( i \) denotes the inclusion \( i(u) = u \otimes 1 \) and \( \pi \) is the projection. The hypothesis that \( p_\# : G_{2r+1}(E,X;j) \otimes \mathbb{Q} \to \pi_2(S^{2r+1}) \otimes \mathbb{Q} \) is onto—included in (2), when translated into our derivation setting, gives the existence of a \( \pi \)-derivation \( \psi \in \operatorname{Der}_{2r+1}(\Lambda(u) \otimes \Lambda V, \Lambda V; \pi) \) that is a cocycle, and that satisfies \( \psi(u) = 1 \). Using this \( \psi \), define a linear map \( \Phi : \Lambda(u) \otimes \Lambda V \to \Lambda(u) \otimes \Lambda V \) by setting \( \Phi(a+ub) = a+ub+u\psi(a) \) for a typical element \( a+ub \in \Lambda(u) \otimes \Lambda V \). We will check that \( \Phi \) is actually a DG algebra isomorphism \( \Lambda(u) \otimes \Lambda V, D \to (\Lambda(u) \otimes \Lambda V, d) \). First, \( \Phi \) is an algebra map. This follows from the fact that \( \psi \) is a derivation. Suppose given two elements \( a+ub, a'+ub' \in \Lambda(u) \otimes \Lambda V \). Then we have

\[
\Phi((a+ub)(a'+ub')) = \Phi(aa' + u((-1)^{|a|}ab'+ba')) \\
= aa' + u((-1)^{|a|}ab' + ba') + u\psi(aa').
\]

On the other hand, we have

\[
\Phi(a+ub)\Phi(a'+ub') = (a+ub + u\psi(a))(a'+ub' + u\psi(a')) \\
= aa' + u((-1)^{|a|}ab' + ba' + (-1)^{|a|}a\psi(a') + \psi(a)a').
\]

These agree, since \( \psi(aa') = \psi(a)a' + (-1)^{|a|}a\psi(a') \). Next, we check that \( \Phi \) commutes with differentials, that is, that \( \Phi D = d\Phi \). This will follow from the fact that \( \psi \) is a cocycle. First observe that \( d\Phi(u) = 0 = \Phi D(u) \). It is thus sufficient to check that \( \Phi D(\chi) = d\Phi(\chi) \) for a typical element \( \chi \in \Lambda V \). To this end, write \( D(\chi) = d(\chi) + u\theta(\chi) \). Now we calculate

\[
\Phi D(\chi) = \Phi(d(\chi) + u\theta(\chi)) = d(\chi) + u\theta(\chi) + u\psi(d\chi)
\]

and

\[
d\Phi(\chi) = d(\chi + u\psi(\chi)) = d(\chi) - ud\psi(\chi).
\]

These agree since \( \psi \) is a cocycle, whence \( -ud\psi(\chi) = \psi D(\chi) = \psi(d(\chi) + u\theta(\chi)) \).

It is evident that \( \Phi \) is an isomorphism, since we have \( \Phi(u) = u \) and \( \pi \circ \Phi = \pi \).

Therefore, \( \Phi : (\Lambda(u) \otimes \Lambda V, D) \to (\Lambda(u) \otimes \Lambda V, d) \) is a DG isomorphism and the fibration is rationally trivial. \( \square \)
This leads to the following equivalent phrasing of Conjecture 2.10:

**Corollary 4.10.** Let $X$ be an $F_0$-space. Then $X$ satisfies Conjecture 2.10 if and only if the $G$-sequence of the fibre inclusion in every fibration of the form $X \to E \to S^{2n+1}$ decomposes into split short exact sequences as in (2) of Theorem 4.9.

**Proof.** In [20], Meier showed that Halperin’s conjecture for $X$ is equivalent to the collapsing of the rational Serre spectral sequence for all fibrations with fibre $X$ and base an odd sphere. By [17, Th.2.3] this latter condition is equivalent to the rational homotopy triviality of the fibration. The result now follows from Theorem 4.9. □

**Appendix A**

In this appendix, we give a careful justification of a result from DG algebra homotopy theory that is used in a crucial way to establish Theorem 2.1. Since it is a technical appendix, we rely on a greater degree of familiarity with techniques from rational homotopy theory. We use the notion of pullback in the DG algebra setting. By this, we mean the following. Suppose given DG algebra maps $f : A \to C$ and $g : B \to C$. Then we form the (DG algebra) pullback (or fibre product, as it is called in [3]) as $A \otimes_C B = \{ (x, y) \in A \oplus B \mid f(x) = g(y) \}$. Here $A \oplus B$ denotes the direct sum of DG algebras. Together with the projections, the pullback forms the following (strictly) commutative square of DG algebra maps:

$$
\begin{array}{ccc}
A \oplus_C B & \xrightarrow{p_1} & A \\
\downarrow{p_2} & & \downarrow{f} \\
B & \xrightarrow{g} & C
\end{array}
$$

This square possesses the usual universal property of pullbacks. Namely, suppose given DG algebra maps $\alpha : Z \to A$ and $\beta : Z \to B$ that satisfy $f \circ \alpha = g \circ \beta$. Then there exists a DG algebra map $\phi = (\alpha, \beta) : Z \to A \oplus_C B$, which is the unique DG algebra map for which $p_1 \circ \phi = \alpha$ and $p_2 \circ \phi = \beta$. We emphasize that throughout this appendix we distinguish carefully between diagrams that are strictly commutative and ones that are commutative only up to DG homotopy. Indeed, it is precisely this distinction that calls for the proofs of this appendix.

The following basic property of the pullback is readily gleaned from the discussion in [3, Sec.13(a)]:

**Lemma A.1.** Suppose that either $f$ or $g$ is surjective in the pullback diagram (13). If $f$ is a quasi-isomorphism, then $p_2$ is a quasi-isomorphism.

We also use the so-called surjective trick, described in [3, Sec.12(b)]. Given a DG algebra map $\eta : B \to A$, this manoeuvre results in a diagram

$$
\begin{array}{ccc}
B & \xrightarrow{\lambda} & B \otimes E(A) \\
\downarrow{1 \cdot \varepsilon} & & \downarrow{\gamma} \\
A
\end{array}
$$

in which $\gamma$ is a surjection, and both $1 \cdot \varepsilon$ and $\lambda$ are quasi-isomorphisms. Some parts of the diagram commute, thus $(1 \cdot \varepsilon) \circ \lambda = 1$ and $\gamma \circ \lambda = \eta$. Other compositions result in commutativity only up to DG homotopy. Recall that the notion of DG
homotopy is only defined for DG algebra maps from a minimal model. Given any map \( \phi: M \to B \otimes E(A) \) from a minimal model into \( B \otimes E(A) \), we have \( \phi \sim \lambda \circ (1 \cdot \varepsilon) \circ \phi \), where \( \sim \) denotes DG homotopy of maps from a minimal model.

In particular, we thus have \( \eta \circ (1 \cdot \varepsilon) \circ \phi \sim \gamma \circ \phi \).

Now suppose given a map \( f: X \to Y \). We choose and fix a minimal model \( M_f: M_Y \to M_X \) for \( f \) as follows ([3, Sec.12(c)]): Let \( A^*(f): A^*(Y) \to A^*(X) \) denote the map induced by \( f \) on polynomial differential forms. Let \( \eta_X: M_X \to A^*(X) \) and \( \eta_Y: M_Y \to A^*(Y) \) denote minimal models for \( X \) and \( Y \). As in [3, Sec.12(b)], we convert \( \eta_X \) into a surjection \( \gamma_X: M_X \otimes E(A^*(X)) \to A^*(X) \) and lift \( A^*(f) \circ \eta_Y \) through the surjective quasi-isomorphism \( \gamma_X \), using [3, Lem.12.4], to obtain \( \phi_f: M_Y \to M_X \otimes E(A^*(X)) \). Now set \( M_f = \gamma_X \circ \phi_f \). All this is summarized in the following diagram.

In this and subsequent diagrams, we indicate that a map is a quasi-isomorphism with the symbol \( \simeq \). By construction, we have \( \gamma_X \circ \phi_f = A^*(f) \circ \eta_Y \), \( \gamma_X \circ \alpha = \eta_X \), and \( \beta \circ \alpha = 1 \). Remaining parts of the diagram only commute up to DG homotopy, however, thus we have \( \eta_X \circ \beta \sim \gamma_X \), \( \eta_X \circ M_f \sim A^*(f) \circ \eta_Y \), and so on.

Now let \( \alpha: S^n \to \text{map}(X,Y;f) \) be a representative of a homotopy class in \( \pi_n(\text{map}(X,Y;f)) \). Let \( F: S^n \times X \to Y \) be the adjoint map for \( \alpha \), that is, \( F(s,x) = \alpha(s)(x) \). Since \( \alpha \) is a based map, we have \( F \circ i = f: X \to Y \), where \( i: X \to S^n \times X \) denotes (based) inclusion into the second summand \( i(x) = (*,x) \).

In the following result, we justify that the Sullivan minimal model of any adjoint map, and a DG homotopy between two such, have the restricted form that we require of them for the definition and well defined-ness of \( \Phi_f \).

**Proposition A.2.** Suppose given maps \( F,G: S^n \times X \to Y \) and \( H: S^n \times X \times I \to Y \) a homotopy from \( F \) to \( G \) that is stationary on \( X \times I \). Suppose that \( H(*,x,t) = f(x) \) for \( f: X \to Y \) and let \( M_f: M_Y \to M_X \) be a fixed choice of minimal model for \( f \). There is a DG homotopy \( \mathcal{H}: M_Y \to M_{S^n} \otimes M_X \otimes \Lambda(t,dt) \) from \( M_F \) to \( M_G \), minimal models for \( F \) and \( G \), of the form

\[
\mathcal{H}(\chi) = 1 \otimes M_f(\chi) \otimes 1 + \text{terms in } (M_{S^n})^+ \otimes M_X \otimes \Lambda(t,dt).
\]

In particular, any map \( F: S^n \times X \to Y \) that satisfies \( F \circ i = f \) has a minimal model \( M_F: M_Y \to M_{S^n} \otimes M_X \) of the form

\[
M_F(\chi) = 1 \otimes M_f(\chi) + \text{terms in } (M_{S^n})^+ \otimes M_X.
\]
Proof. Construct the following pullback:

\[
\begin{array}{ccc}
P & \xrightarrow{p_1} & M_X \otimes \Lambda(t, dt) \otimes E(A^*(X \times I)) \\
p_2 & & \downarrow \cong \\
A^*(S^n \times X \times I) & \xrightarrow{A^*(i)} & A^*(X \times I)
\end{array}
\]

Here, \(i : X \times I \to S^n \times X \times I\) denotes the inclusion \(i(x,t) = (x,x,t)\) and \(\gamma\) denotes the surjective quasi-isomorphism obtained by converting the quasi-isomorphism \(M_X \otimes \Lambda(t, dt) \to A^*(X \times I)\) to a surjection. Since \(i\) is an inclusion, the induced map \(A^*(i)\) is a surjection. From Lemma A.1, we have that \(p_2\) is a quasi-isomorphism. Now consider the following commutative diagram:

\[
\begin{array}{ccc}
M_{S^n} \otimes M_X \otimes \Lambda(t, dt) \otimes E(A^*(S^n \times X)) & \xrightarrow{\phi_H} & M_Y \\
\downarrow \quad \gamma' & \quad \downarrow \gamma & \\
A^*(S^n \times X \times I) & \xrightarrow{A^*(i)} & A^*(X \times I)
\end{array}
\]

Here, \(\gamma'\) is the surjective quasi-isomorphism obtained by converting the quasi-isomorphism \(M_{S^n} \otimes M_X \otimes \Lambda(t, dt) \to A^*(S^n \times X \times I)\) to a surjection, \(\phi_f : M_Y \to M_X \otimes E(A^*(X))\) denotes the lift used above to obtain our fixed choice of minimal model for \(f\), and \(J : M_X \to M_Y \otimes \Lambda(t, dt)\) denotes the map \(J(x) = x \otimes 1\). From the pullback, we obtain the maps \(\phi\) and \(\psi\) indicated. Since both \(p_2\) and \(\gamma'\) are quasi-isomorphisms, it follows that \(\phi\) is a quasi-isomorphism. We claim that \(\phi\) is also surjective. For suppose \((a,b) \in P\), so that \(\gamma(a) = A^*(i)(b)\). Since \(\varepsilon \cdot (1 \otimes 1 \otimes E(A^*(i)))\) is surjective, we can pick \(x \in M_{S^n} \otimes M_X \otimes \Lambda(t, dt) \otimes E(A^*(S^n \times X \times I))\) with \(\phi(x) = (a,b')\), and \(A^*(i)(b') = \gamma(a) = A^*(i)(b)\). Thus \(b - b' \in \ker A^*(i)\). So now pick \(y \in E(A^*(S^n \times X \times I))\) with \(\gamma_{S^n} (1 \otimes 1 \otimes 1 \otimes y) = b - b'\)—recall that \(E(A^*(S^n \times X \times I))\) is freely generated by the vector space \(A^*(S^n \times X \times I)\) and its suspension. Then \(\phi(x + 1 \otimes 1 \otimes 1 \otimes y) = (a,b)\); so \(\phi\) is indeed surjective.

Now lift \(\psi\) through the surjective quasi-isomorphism \(\phi\), to obtain a map \(\phi_H : M_Y \to M_{S^n} \otimes M_X \otimes \Lambda(t, dt) \otimes E(A^*(S^n \times X \times I))\) with \(\phi \circ \phi_H = \psi\). As usual for the minimal model of a map, we now set \(H = \beta \circ \phi_H : M_Y \to M_{S^n} \otimes M_X \otimes \Lambda(t, dt)\).

We now check that this DG homotopy starts and ends at minimal models for \(F\) and \(G\) respectively. Let \(j_0 : S^n \times X \to S^n \times X \times I\) denote inclusion at \(t = 0\), and \(\varepsilon_0 : \Lambda(t, dt) \to \mathbb{Q}\) the augmentation given by \(\varepsilon_0(t) = 0\) and \(\varepsilon_0(dt) = 0\). We must check that \((1 \otimes 1) \cdot \varepsilon_0 : M_{S^n} \otimes M_X \otimes \Lambda(t, dt) \to M_{S^n} \otimes M_X\) is a minimal model.
for $F$. For this, consider the following diagram:

$$
\begin{array}{ccc}
\mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt) \otimes E(A^*(S^n \times X \times I)) & \xrightarrow{\phi_H} & \mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes E(A^*(S^n \times X)) \\
\mathcal{M}_Y & \xrightarrow{\eta_Y} & \mathcal{M}_{S^n} \otimes \mathcal{M}_X \otimes \Lambda(t, dt) \\
A^*(Y) & \xrightarrow{A^*(H)} & A^*(S^n \times X) \\
A^*(j_0) & \xrightarrow{A^*(f_0)} & A^*(S^n \times X)
\end{array}
$$

We have

$$
\eta'' \circ (1 \otimes 1) \cdot \varepsilon_0 \circ \mathcal{H} = A^*(j_0) \circ \eta' \circ \beta' \circ \phi_H
$$

$$
\sim A^*(j_0) \circ \gamma' \circ \phi_H = A^*(j_0) \circ A^*(H) \circ \eta_Y
$$

$$
= A^*(F) \circ \eta_Y.
$$

It follows that $(1 \otimes 1) \cdot \varepsilon_0 \circ \mathcal{H}$ is a minimal model for $F$. A similar argument shows that $(1 \otimes 1) \cdot \varepsilon_1 \circ \mathcal{H}$ is a minimal model for $G$, where $\varepsilon_1 : \Lambda(t, dt) \rightarrow \mathbb{Q}$ is the augmentation given by $\varepsilon_1(t) = 1$ and $\varepsilon_1(dt) = 0$.  

References


[18] G. Lupton and S. B. Smith, Rationalized evaluation subgroups II: Quillen models and adjoint maps, Pre-print

[19] Rank of the fundamental group of a component of a function space, Pre-print


