Notes on center of mass

Center of gravity (similar but easier to grasp intuitively)

Setting the stage

All objects in the xy-plane, force of gravity points in negative z-direction. Say we have four objects as shown:

1. 2-pound object @ (-5, -1)
2. 11-lb. object @ (-3, 5)
3. 4-lb object @ (1, -2)
4. 3-lb object @ (2, 3)

Imagine them rigidly joined into a whole.

If you attempt to balance the whole on the line x=-2, will it balance? If not, which way will it tip?

The 2-lb. object is 3 feet to left, so will pull 5 with "torque" of 6 pound-feet.

We will let i→negative and j→positive, so the 2-lb. object contributes torque of (-6). Do this for all the objects:

Total is 7 lb-feet (see figure).

So, answers to questions are:

No, it will not balance
It will tip 2 (down to right)

For any line x=a:

1. Total torque > 0: tips 2
total torque < 0: tips 6
2. total torque = 0: does not tip (balances)
Note that for weight \( w \) at \((x_i, y_i)\), torque around \( x = a \) (including sign!) is very conveniently equal to \((x-a)w_i\); so if you have weight \( w_i \) at point \((x_i, y_i)\) \((1 \leq i \leq n)\), total torque around line \( x = a \) is:

\[
\sum_{i=1}^{n} (x_i - a)w_i
\]

Switch to center of mass (work with mass not weight, so independent of gravity; "torque" in this context is called "moment"), but calculations work the same way. For the same four objects, say now that the given weights are instead their masses.

In figure at left, I have calculated total moment around the line \( x = 0 \) (ie the \( y \)-axis); total = \(-33 \text{ (cm-gm.oz)}\) something. Moment around the \( y \)-axis is denoted "\( M_y \)". So in this case:

\[
M_y = -33
\]

Calculations for continuous objects in \( xy \)-plane ("lamina")

Setting the stage:

Object occupies planar region \( R \). At point \((x, y)\) in \( R \), object has (mass-) density \( p(x, y) \text{ gm/cm}^2\), so "infinitesimal" mass at \((x, y)\) is \( p(x, y) dA \), and moment at \((x, y)\) is \((x-a)p(x, y)dA \).

Total moment around line \( x = a \) is:

\[
\int \int_R (x-a)p(x, y)dA
\]
Center of mass for lines parallel to the y-axis

The next task is to find out which lines \( x = a \) (for any) have the property that object will balance on them:

\[
\sum_{i=1}^{n} (x_i - a) m_i = 0 \quad \text{or} \quad \int_R (x-a) \rho(x,y) \, dA = 0.
\]

There is a unique solution in either case. Calculations are parallel; I will do them only for the continuous case.

\[
\int_R (x-a) \rho(x,y) \, dA = 0 \quad \text{(solve for } a)\]

Multiply out:
\[
\int_R x \rho(x,y) - a \rho(x,y) \, dA = 0
\]

Distribute integral:
\[
\int_R x \rho(x,y) \, dA - a \int_R \rho(x,y) \, dA = 0
\]

Transpose:
\[
\int_R x \rho(x,y) \, dA = a \int_R \rho(x,y) \, dA
\]

Divide:
\[
a = \frac{\int_R x \rho(x,y) \, dA}{\int_R \rho(x,y) \, dA} = \frac{M_y}{\text{mass}}
\]
The solution that we found is called the center of mass in the x-direction and is denoted $\bar{x}$:

$$\bar{x} = \frac{\int \int x \rho(x,y) \, dA}{\int \int \rho(x,y) \, dA} = \frac{M_y}{\text{mass}}; \text{it has the property that } \int \int (x-x) \rho(x,y) \, dA = 0$$

Similarly, the solution to $\sum_{i=1}^{n} (x_i - a) m_i = 0$ is

$$\sum_{i=1}^{n} \frac{x_i m_i}{m_i} = \frac{M_y}{\text{mass}}; \text{denoted } \bar{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i} = \frac{M_y}{\text{mass}}$$

$\bar{x}$ satisfies: $\sum_{i=1}^{n} (x_i - \bar{x}) m_i = 0$

For the four objects, $\bar{x} = \frac{M_y}{\text{mass}} = \frac{-33}{20} = -1.65$.

In the figure, I have directly calculated $\sum_{i=1}^{4} (x_i - (-1.65)) m_i$ and verified that it equals zero.

When you reverse the roles of the axes:

Everything works the same way:

Moment around $y = b$ is $\sum_{i=1}^{n} (y_i - b) m_i$ or $\int \int (y-b) \rho(x,y) \, dA$

This comes out to zero for

$$b = \bar{y}$$

(center of mass in y-direction)

$$\bar{y} = \frac{\int \int y \rho(x,y) \, dA}{\int \int \rho(x,y) \, dA} = \frac{M_x}{\text{mass}}$$

(1.4)
Overall center of mass

Summary: a lamina (or collection of objects) will balance on the line \( x = \bar{x} \) and (separately) on the line \( y = \bar{y} \). These lines meet at the point \((\bar{x}, \bar{y})\).

The point \((\bar{x}, \bar{y})\) is the (overall) center of mass of the lamina/collection of objects.

Reasonable question: will the lamina/object balance on the point \((\bar{x}, \bar{y})\)? In other words, could you balance the thing on your finger-tip? The answer is yes.

Here's why: If it did not balance, then it would have to tip in some direction, which would mean that it would not balance on the line \( y = \bar{y} \) (through \((\bar{x}, \bar{y})\), perpendicular to direction \( \ell \) of the tipping.

On the next page, I will show that this cannot happen, because the lamina will have to balance on every line that passes through \((\bar{x}, \bar{y})\).

(A similar argument can be made to show the corresponding property for the center of mass of a system of objects.)
A lamina will balance on its center of mass.

Let a lamina occupying region \( D \) in the \( xy \)-plane have mass density \( \rho(x, y) \). As discussed in class, the lamina can be balanced on either a horizontal or a vertical line that passes through \((\bar{x}, \bar{y})\), where

\[
\bar{x} := \int_{D} \frac{x \rho \, dA}{\int_{D} \rho \, dA}
\]

and

\[
\bar{y} := \int_{D} \frac{y \rho \, dA}{\int_{D} \rho \, dA}.
\]

This handout will prove that the lamina will also balance on any other line through \( (\bar{x}, \bar{y}) \). Let \( \ell \) be the line through \( (\bar{x}, \bar{y}) \) and perpendicular to a given nonzero vector \( (A, B) \). For any point \( (x, y) \) in the lamina, let \( d(x, y) \) be the perpendicular distance from \( (x, y) \) to \( \ell \), where "+" is used if \( (x, y) \) is on the same side of \( \ell \) as \( (A, B) \), and where "−" is used if \( (x, y) \) is on the opposite side of \( \ell \) from \( (A, B) \). Then, the lamina will balance on the line \( \ell \) precisely if the moments on both sides of \( \ell \) cancel each other out—that is, if

\[
\int_{D} d(x, y) \rho(x, y) \, dA = 0
\]

comes out to zero. This is exactly what I will prove. First, notice that \( d(x, y) \) is precisely

\[
\text{comp}_{(A, B)} (x - \bar{x}, y - \bar{y}),
\]

where \( \text{comp}_{(A, B)} (\vec{b}) \) is the signed length of the projection of vector \( \vec{b} \) onto vector \( \vec{a} \) (see p.783 of the text). Thus

\[
d(x, y) = \text{comp}_{(A, B)} (x - \bar{x}, y - \bar{y}) = \frac{(A, B) \cdot (x - \bar{x}, y - \bar{y})}{\| (A, B) \|} = \frac{A}{\| (A, B) \|} (x - \bar{x}) + \frac{B}{\| (A, B) \|} (y - \bar{y}),
\]

so

\[
\int_{D} d(x, y) \rho(x, y) \, dA = \int_{D} \left[ \frac{A}{\| (A, B) \|} (x - \bar{x}) + \frac{B}{\| (A, B) \|} (y - \bar{y}) \right] \rho(x, y) \, dA
\]

\[
= \int_{D} \frac{A}{\| (A, B) \|} (x - \bar{x}) \rho(x, y) \, dA + \int_{D} \frac{B}{\| (A, B) \|} (y - \bar{y}) \rho(x, y) \, dA
\]

\[
= \frac{A}{\| (A, B) \|} \int_{D} (x - \bar{x}) \rho(x, y) \, dA + \frac{B}{\| (A, B) \|} \int_{D} (y - \bar{y}) \rho(x, y) \, dA.
\]

But note that integral \((a)\) is the moment around the line \( x = \bar{x} \), which is zero.\(^1\) By the same token, \( \bar{y} \) is the number that forces integral \((b)\) to be zero. Thus \((1)=(2)=0\), and the result is proved. \(\Box\)

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\(^1\) Recall that we found \( \bar{x} \) by setting integral \((a)\) equal to zero and solving for \( \bar{x} \).