Velocity, Curvature and Acceleration

1 Introduction.

The related body of ideas discussed in this handout are all covered in §13.3 and §13.4; I prepared this handout in order to explain my approach to these ideas, which I prefer to the approach taken in the text.\footnote{For example: by introducing formula (17) (handout, p.6)—which appears on p.856 of the text—before introducing formula (24) (handout, p.9)—which the text does not introduce until p.866—I get a simple, clear derivation of (24) from (17). The text’s derivation of (24) is neither simple nor clear.} Throughout, I will assume

1. that \( t \mapsto \vec{r}(t) \) is twice differentiable, and
2. that \( \vec{r}''(t) \) is never zero—that is, that the object under discussion is in motion at all times.\footnote{The second assumption is not strictly necessary, but it simplifies the discussion.}

We will have need of the following observation.

**Observation 1** Let \( t \mapsto \vec{f}(t) \) be differentiable. Then

\[
\| \vec{f}(t) \| \text{ is constant } \iff \vec{f}(t) \cdot \vec{f}'(t) \equiv 0 \text{ for all } t.
\]  

**Proof.**

\[
\| \vec{f}(t) \| \text{ is constant } \iff \left( \| \vec{f}(t) \| \right)^2 \text{ is constant}
\]

\[
\iff \vec{f}(t) \cdot \vec{f}(t) \text{ is constant}
\]

[Calc I: \( h(t) \equiv c \iff h'(t) \equiv 0 \) \( \iff \frac{d}{dt} \left[ \vec{f}(t) \cdot \vec{f}(t) \right] \equiv 0 \)]

[Rule #4 on p.850 for \( \frac{d}{dt} (\vec{u}(t) \cdot \vec{v}(t)) \) \( \iff \left( \vec{f}(t) \cdot \vec{f}'(t) \right) + \left( \vec{f}'(t) \cdot \vec{f}(t) \right) \equiv 0 \)

\( \iff 2 \left( \vec{f}(t) \cdot \vec{f}'(t) \right) \equiv 0 \)

\( \iff \vec{f}(t) \cdot \vec{f}'(t) \equiv 0. \)

2 Arclength as Parameter.

For any \( t \geq t_0 \), let \( s(t) = \int_{t_0}^{t} \| \vec{r}'(u) \| \, du \); that is, if you think of \( \vec{r}(t) \) as the position of a moving car at time \( t \), then \( s(t) \) is the distance traveled—the amount by which the car’s odometer reading
will have increased over the time interval \([t_0, t]\). As we have seen, the speed at time \(t\)—the rate of change of the odometer reading—equals the length of the velocity vector:

\[
s'(t) = \| \vec{r}'(t) \| \text{("speed(t)")}. \tag{2}
\]

It is sometimes useful to express position, velocity, speed, etc. as functions of odometer reading rather than as functions of time, because one can then bring the Chain Rule\(^3\) to bear. The first instance is the function \(s \mapsto r(t)(s)\), which expresses position as a function of the odometer reading, is related to the function \(t \mapsto \vec{r}(t)\) through the equation

\[
\vec{r}(t) = r(t)(s(t)), \tag{3}
\]

to which I will need to apply the Chain Rule (see equation (4) below).

Throughout this handout, I use notation like that in equation (3) to indicate the introduction of the odometer readings into a formula.\(^4\)

### 3 The Unit Tangent and Its Derivative

Differentiating equation (3) gives

\[
\vec{r}'(t) = \frac{d}{dt} \left[ r(t)(s(t)) \right]
\]

(Chain Rule)

\[
= s'(t) r'(t)(s(t))
\]

\[
= \| \vec{r}'(t) \| r'(t)(s(t)). \tag{4}
\]

and dividing through by \(\| \vec{r}'(t) \|\) then gives

\[
\frac{1}{\| \vec{r}'(t) \|} \vec{r}'(t) = r'(t)(s(t)).
\]

This tells us that \(r'(t)(s(t))\) is the unit tangent \(\vec{T}(t)\), so that from equation (5),

\[
\vec{r}'(t) = \| \vec{r}'(t) \| \vec{T}(t) = \text{speed}(t) \vec{T}(t). \tag{6}
\]

Below, equation (6) will prove useful for the analysis of acceleration.

A vector of particular interest is \(\vec{T}'(t)\), the derivative of the unit tangent vector \(\vec{T}(t)\). To understand \(\vec{T}'(t)\) clearly, it will help to maintain two mental images of \(\vec{T}(t)\):

\(^3\)text, p. 850, Rule #6

\(^4\)It is worth pointing out that there may be more than one time \(t\) for a given odometer reading \(s\)—this will happen if the object sits in one position without moving for a while—but the odometer functions will still make sense. For example, a parked car has the same odometer reading \(s\) for many times \(t\), but it also has the same position at all of those times. In other words, the value of \(s\) completely determines the position of the car.
View from the curve. From this perspective, \( \vec{T}(t) \) is a unit vector tangent to the curve at \( \vec{r}(t) \), pointing in the direction of motion; as \( t \) changes, \( \vec{T}(t) \) slides along the curve.

\[ \vec{T}'(t) \]

\[ \vec{T}(t) \]

View from the origin. \( \vec{T}(t) \) is drawn in standard position the head of the vector will stay on the unit circle (in the \( \mathbb{R}^2 \) case) or sphere (in the \( \mathbb{R}^3 \) case); so that \( \vec{T}(t) \) may be thought of as the position at time \( t \) of a different car —let’s dub this car the unitcar—that is driving along the unit circle (or respectively, driving around on the unit sphere). The vector \( \vec{T}'(t) \) can then be viewed as the unitcar’s velocity at time \( t \).

\[ \vec{T}'(t) \]

\[ \vec{T}(t) \]

We will examine both the magnitude and direction of the vector \( \vec{T}'(t) \); each of them supplies important information.

Direction. Since \( \|\vec{T}(t)\| = 1 \), it follows from observation 1 that \( \vec{T}'(t) \cdot \vec{T}(t) = 0 \), so that the vector \( \vec{T}'(t) \) (whenever nonzero) is perpendicular to \( \vec{T}(t) \). Since \( \vec{T}'(t) \) is the velocity vector for the unitcar, it points in the direction of motion of the unitcar, or (equivalently) the direction in which \( \vec{T}(t) \) is turning. For the view from the curve, this says the following:

The vector \( \vec{T}'(t) \) is perpendicular to the curve at \( \vec{r}(t) \), and it points in the direction in which \( \vec{T}(t) \) (and hence also \( r'(t) \)) are turning. In the \( \mathbb{R}^2 \) case, this is the concave side of the curve.

Magnitude. Since \( \vec{T}'(t) \) is the unitcar’s velocity vector, \( \|\vec{T}'(t)\| \) is the speed with which \( \vec{T}(t) \) is moving (in distance-per-unit-time units: mi./hr., ft./sec., etc.). In the \( \mathbb{R}^2 \) case, this is also the speed with which the angle \( \theta(t) \) is changing (in radians-per-unit-time).\(^5\) Incorporating this information into the \( r(t) \) view leads to the conclusion that

\(^5\)the angle between \( \vec{T}(t) \) and the positive x-axis, familiar from polar coordinates

\(^6\)In the \( \mathbb{R}^3 \) case, one can also view \( \|\vec{T}'(t)\| \) as speed of angle change, but the angle \( \theta(t) \) must be defined in a plane parallel to the plane \( P \) (equation (15) on p.6).
$$\|\vec{T}'(t)\|$$ is the speed, in radians-per-unit-time, with which the direction (of motion) is changing at time \( t \). One sometimes writes:

$$\|\vec{T}'(t)\| = \left|\frac{d\theta(t)}{dt}\right|.$$ (7)

3.1 Curvature.

Suppose there are two cars, car \(_1\) and car \(_2\), driving along the same track, and say that car \(_1\) is going twice as fast as car \(_2\). Suppose also that we have stationed two observers at two points \( P_a \) and \( P_b \) along the track. Let

- \( t_1^a \) be the time at which car \(_1\) passes \( P_a \).
- \( t_1^b \) be the time at which car \(_2\) passes \( P_a \).
- \( t_2^a \) be the time at which car \(_1\) passes \( P_b \), and
- \( t_2^b \) be the time at which car \(_2\) passes \( P_b \).

Clearly, due to the difference in the speeds, each observer is going to observe the rate at which car \(_1\)’s direction is changing to be twice the rate at which car \(_2\)’s direction is changing:

$$\|\vec{T}_1'(t_1^b)\| = 2\|\vec{T}_2'(t_2^b)\|,$$

and

$$\|\vec{T}_1'(t_2^a)\| = 2\|\vec{T}_2'(t_2^b)\|.$$ (8)

Similarly, at either \( P_a \) or \( P_b \): if car \(_1\) is going \( \left\{\begin{array}{c} 3 \text{ times} \\ 4 \text{ times} \\ x \text{ times} \end{array}\right\} \) as fast as that of car \(_2\), then car \(_1\)’s rate of direction change will be \( \left\{\begin{array}{c} 3 \text{ times} \\ 4 \text{ times} \\ x \text{ times} \end{array}\right\} \) as fast as car \(_2\). More generally: at \( P_a \) or \( P_b \) (or at any other point), the ratio of speed of direction change to speed of motion will be the same for all cars. In other words:

At any point \( P \) on the curve, there is a constant \( \kappa_r \) with the property that

$$\vec{r}(t) = P \implies \|\vec{T}'(t)\| = \kappa_r\|\vec{r}'(t)\|.$$ (9)

Let us now examine what \( \kappa_r \) tells us about \( P \). Suppose that car \(_1\) is travelling at 60mph when it passes through both \( P_a \) and \( P_b \) and that the road is sharply curved at \( P_a \) but almost straight at \( P_b \). We would expect \( \|\vec{T}_1'(t_1^b)\| \) to be larger than \( \|\vec{T}_1'(t_1^a)\| \): the 60mph speed causes a faster direction change at \( P_a \), because a small distance forward results in a larger direction change there. The size of \( \kappa_r \) in (9) thus measures how straight or curved the road is at \( p \). One thus finds the curviness at a point \( P \) on the curve by solving (9) for \( \kappa_r \):

\(^7\)Indeed, if \( P_b \) occurs along a completely straight stretch of road, then car \(_1\)’s direction is not changing at all at time \( t_1^b \), so \( \|\vec{T}_1'(t_1^b)\| = 0 \).
Definition 1 The curvature $\kappa(t)$ of curve $s \mapsto \vec{r}(s)$ at the point $P = \vec{r}(t)$ is the ratio

$$\kappa(t) := \frac{\|\vec{T}'(t)\|}{\|\vec{T}(t)\|} = \frac{\|\vec{T}'(t)\|}{\text{speed}(t)}.$$ (10)

As a definition, equation (10) is straightforward, but it usually leads to very messy calculations.\(^8\) Fortunately, there are better ways to calculate $\kappa(t)$ (see equations (19) and (24) below).

If we write $\vec{T}(t)$ as $\vec{T}_A(s(t))$—where $s \mapsto \vec{T}_A(s)$ gives the unit tangent as a function of odometer reading—then

$$\vec{T}'(t) = \frac{d}{dt} \left[ \vec{T}_A(s(t)) \right]$$

(Chain Rule) $$= s'(t) \left( \vec{T}_A'(s(t)) \right)$$

$$\vec{T}'(t) = \text{speed}(t) \left( \vec{T}_A'(s(t)) \right).$$ (11)

Equating magnitudes in (11) then gives

$$\|\vec{T}'(t)\| = \text{speed}(t) \|\vec{T}_A'(s(t))\|,$$ (12)

from which, by substituting (12) into (10), we see that

$$\kappa(t) = \|\vec{T}_A'(s(t))\|.$$ (13)

Equation (13) can be interpreted to mean that

| The curvature at point $P = \vec{r}(t)$ is the rate with respect to arclength with which direction is changing at $P$. |

3.2 Coordinates at $\vec{r}(t)$.

At times $t$ for which $\vec{T}'(t) \neq \vec{0}$, it is useful to attach three mutually perpendicular unit vectors to the curve at $\vec{r}(t)$, which can be visualized of as a sort of alternate coordinate system that slides along the curve as $t$ changes.

1. The first of these, as you most likely have guessed, is the unit tangent vector $\vec{T}(t)$.

2. The second vector, denoted $\vec{N}(t)$ and called the principal normal to the curve at $\vec{r}(t)$, is the unit vector that goes in the same direction as does $\vec{T}'(t)$. A routine calculation from equations (12) and (13) shows that

$$\vec{N}(t) := \left( \frac{1}{\kappa(t)} \right) \vec{T}_A'(s(t)) = \left( \frac{1}{\text{speed}(t)\kappa(t)} \right) \vec{T}'(t);$$ (14)

\(^8\)Formula (10) makes a mess except in the case that the object is traveling at a constant rate of speed.
3. The third vector, denoted $\vec{B}(t)$ and called the binormal vector at $\vec{r}(t)$, is the vector

$$\vec{B}(t) := \vec{T}(t) \times \vec{N}(t).$$

This vector is used principally to construct the equation of the following plane:

$$(\text{Plane } \mathcal{P}) \quad \vec{B}(t) \cdot (\vec{x} - \vec{r}(t)) = 0. \quad (15)$$

This plane is called the osculating plane of the curve at $\vec{r}(t)$.

- In the $\mathbb{R}^2$ case, $\vec{B}(t)$ makes sense after one has added a $z = 0$ coordinate. In this case, $\vec{B}(t) = \pm \vec{k}$, and equation (15) reduces to $z = 0$, the equation of the $xy$-plane (see exercise 4 on p.6).
- In the $\mathbb{R}^3$ case, as will become clear below, $\mathcal{P}$ can be thought of as "(instantaneous) plane of motion at the point $\vec{r}(t)$.

**Exercise 1** Let $\vec{r}_1$ be defined for $-\infty < t < \infty$, and let $\vec{r}_2(t) = \vec{r}_1(-t)$ (so that $\vec{r}_1$ and $\vec{r}_2$ parametrize the same curve but in opposite directions).

(a): What is the relationship between $\vec{r}_2'(t)$ and $\vec{r}_1'(-t)$?

(b): What is the relationship between $\vec{T}_2(t)$ and $\vec{T}_1(-t)$?

(c): What is the relationship between $\vec{T}_2'(t)$ and $\vec{T}_1'(-t)$?

4 Acceleration.

The derivative of velocity is acceleration:

$$\vec{r}''(t) = \vec{v}'(t) = \vec{a}(t).$$

Roughly speaking, the vector $\vec{a}(t)$ brings about short-term changes to $\vec{r}''(t)$, which is to say that $\vec{a}(t)$ affects both the magnitude of $\vec{r}''(t)$—the speed—and the direction of $\vec{r}''(t)$. There turns out to be a way to tease these two effects apart and to quantify them separately. If we write $\vec{r}''(t) = \text{speed}(t)\vec{T}(t)$ (see equation (6)) and differentiate, we get

$$\vec{a}(t) = \vec{r}''(t) = \frac{d}{dt} \left[ \text{speed}(t)\vec{T}(t) \right].$$

[Rule #3 on p.850 for $\frac{d}{dt} (f(t)\vec{a}(t))$]

$$= \text{speed}'(t)\vec{T}(t) + \text{speed}(t)\vec{T}'(t)$$

[By equation (11)]

$$= \text{speed}'(t)\vec{T}(t) + \text{speed}(t) \left( \text{speed}(t) \left( \vec{T}' \left( \text{speed}(t) \right) \right) \right)$$

[By equation (13)]

$$= \sqrt{a_s(t)} \vec{T}(t) + \sqrt{a_s(t)} \kappa(t) \vec{N}(t) \quad (16)$$

$$\vec{a}(t) = a_s(t)\vec{T}(t) + a_s(t)\vec{N}(t) \quad (17)$$

---

*If $h$ is small, then $\vec{r}''(t + h) - \vec{r}''(t) \approx h\vec{a}(t)$.\*
Equation (17) resolves $\vec{a}(t)$ into the sum of two perpendicular vectors, one a scalar multiple of $\vec{T}(t)$ and the other a scalar multiple of $\vec{N}(t)$. Again, there are two useful mental pictures of these vectors.

**View from the curve.** There is a rectangle $\mathcal{R}(t)$ that has
- one corner at $\vec{r}(t)$,
- one side of length $|a_{\vec{r}}(t)| = a_{\vec{r}}(t)$ can be negative—parallel to the direction of motion,
- one side of length $a_{\vec{N}}(t) = a_{\vec{N}}(t)$ cannot be negative—in the direction of $\vec{N}(t)$, and
- the vector $\vec{a}(t)$ along the diagonal that starts at $\vec{r}(t)$.

As this viewpoint makes clear: since the vectors $\vec{T}(t)$ and $\vec{N}(t)$ are in $\mathcal{P}$, a rectangle $\mathcal{R}(t)$ is in this plane. This is why plane $\mathcal{P}$ is deemed the “plane of motion at the point $\vec{r}(t)$.”

**Exercise 2** Show that the plane defined by (15) contains the vectors $\vec{r}'(t)$ and $\vec{a}(t)$.

**View from the origin.** Restrict your attention to the plane $\mathcal{P}'$ through the origin that is parallel to plane $\mathcal{P}$. In $\mathcal{P}'$, draw $\vec{r}'(t)$ in standard position, and draw the vectors $a_{\vec{r}}(t)\vec{T}(t)$ and $a_{\vec{N}}(t)\vec{N}(t)$ to start at the point $\vec{r}(t)$ (where the vector $\vec{T}'(t)$ ends). This viewpoint helps explain intuitively why one should expect the values of $a_{\vec{r}}(t)$ and $a_{\vec{N}}(t)$ to come out as they did in (16).

**Interpreting $a_{\vec{r}}(t)$.** The length of $a_{\vec{r}}(t)\vec{T}(t)$ (which is parallel to $\vec{T}'(t)$) is the rate at which $\vec{r}'(t)$ is getting longer or shorter; this is why the length of $a_{\vec{r}}(t)\vec{T}(t)$ is the speed $\vec{v}'(t)$.

**Interpreting $a_{\vec{N}}(t)$.** The vector $a_{\vec{N}}(t)\vec{N}(t)$, which is perpendicular to $\vec{T}'(t)$, is in fact tangent to the circle (in $\mathcal{P}'$) of radius $\|\vec{r}'(t)\|$ centered at the origin. Its length is the (instantaneous) linear speed of $\vec{T}'(t)$ around the circle. Now, the linear speed of $\vec{T}'(t)$ around its circle—see box (7) on p. 4—is $\|\vec{T}'(t)\| = \frac{d\theta(t)}{dt}$.

Furthermore, $\vec{T}'(t)$ and $\vec{T}(t)$ are radially aligned, so that they share the same value of $\frac{d\theta(t)}{dt}$. But vector $\vec{T}'(t)$ is in a circle whose radius is $\|\vec{T}'(t)\|$ times the radius of of $\vec{T}'(t)$’s circle, so in order for the $\frac{d\theta(t)}{dt}$ values to match, the linear speed of $\|\vec{T}'(t)\|$ around its circle must be $\|\vec{T}'(t)\|$ times that of $\vec{T}(t)$ around its circle—that is, it must be $\|\vec{T}'(t)\| = a_{\vec{N}}(t)$, so $a_{\vec{r}}(t)$ is the linear speed of $\|\vec{T}'(t)\|$ around its circle.

**Exercise 3** Verify Equation (18).

**Exercise 4** For a curve

$$\vec{r}(t) = (f(t), g(t), 0),$$

show that $\mathcal{P} = \mathcal{P}' = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. (Assume that $\vec{T}'(t) \neq \vec{0}$.) (Hint: there is a way to avoid a lot of messy calculation. Think before you write.)

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*because, clearly, $\vec{b}(t) \cdot \vec{T}(t) = \vec{b}(t) \cdot \vec{N}(t) = 0$. (Recall that a vector $\vec{v}$ is in a given plane $\{ \vec{v} : \vec{n} \cdot (\vec{x} - \vec{x}_0) = 0 \}$ iff $\vec{n} \cdot \vec{v} = 0$, while a point $\vec{x}_1$ is in this plane iff $\vec{n} \cdot (\vec{x}_1 - \vec{x}_0) = 0$.)

*The scalar $a\vec{T}(t)$ is called the tangential component of acceleration, and $a\vec{N}(t)$ is called the normal component of acceleration.
4.1 Computational Consequences of Equation (17).

The fact that the vectors $\vec{T}$ and $\vec{N}$ are perpendicular unit vectors makes it possible to derive several computational short-cuts from equation (17).

(a): Equating the lengths of the two sides in equation (17) and simplifying leads to the relation\(^{11}\)

$$\| \vec{a}(t) \|^2 = a_{\mathcal{R}}(t)^2 + a_{\mathcal{N}}(t)^2,$$

(19)

Because $\| \vec{a}(t) \|^2$ is generally easy to calculate, equation (19) often provides an easy way to calculate one of the numbers $\{a_{\mathcal{R}}(t), a_{\mathcal{N}}(t)\}$ if you know the other.

Exercise 5 Prove equation (19).

(b): Taking the dot product of both sides of (17) with $\vec{a}''(t)$ gives

$$\vec{a}'(t) \cdot \vec{a}''(t) = \vec{a}'(t) \cdot \left( a_{\mathcal{R}}(t)\vec{T}(t) + a_{\mathcal{N}}(t)\vec{N}(t) \right) = \vec{a}'(t) \cdot \left( a_{\mathcal{R}}(t)\vec{T}(t) \right) + \vec{a}'(t) \cdot \left( a_{\mathcal{N}}(t)\vec{N}(t) \right)$$

Because $\vec{a}'(t) \cdot \vec{N}(t) = 0$$

$$= a_{\mathcal{R}}(t) \left( \vec{a}'(t) \cdot \vec{T}(t) \right) = a_{\mathcal{R}}(t) \left( \| \vec{a}'(t) \| \vec{T}(t) \right) \cdot \vec{T}(t) = a_{\mathcal{R}}(t) \| \vec{a}'(t) \| \left( \vec{T}(t) \cdot \vec{T}(t) \right)$$

Because $\vec{T}(t) \cdot \vec{T}(t) = 1$$

$$= a_{\mathcal{R}}(t) \| \vec{a}'(t) \| = a_{\mathcal{R}}(t) \| \vec{a}'(t) \|. \quad (20)$$

Solving for $a_{\mathcal{R}}(t)$ in (20) then gives

$$\text{speed}'(t) = a_{\mathcal{R}}(t) = \frac{\vec{a}'(t) \cdot \vec{a}''(t)}{\| \vec{a}'(t) \|} \quad (21)$$

(c): Taking the cross product of both sides of (17) with $\vec{a}'(t)$ gives

$$\vec{a}'(t) \times \vec{a}''(t) = \vec{a}'(t) \times \left( a_{\mathcal{R}}(t)\vec{T}(t) + a_{\mathcal{N}}(t)\vec{N}(t) \right) = \left( \| \vec{a}'(t) \| \vec{T}(t) \right) \times \left( a_{\mathcal{R}}(t)\vec{T}(t) + a_{\mathcal{N}}(t)\vec{N}(t) \right)$$

Because $\vec{T}(t) \times \vec{T}(t) = 0$$

$$= a_{\mathcal{R}}(t) \| \vec{a}'(t) \| \left( \vec{T}(t) \times \vec{T}(t) \right) + a_{\mathcal{N}}(t) \| \vec{a}'(t) \| \left( \vec{T}(t) \times \vec{N}(t) \right)$$

Equating lengths in (22) and solving for $a_{\mathcal{N}}(t)$ then gives

$$a_{\mathcal{N}}(t) = \| \vec{a}'(t) \|^2 \kappa(t) = \frac{\| \vec{a}'(t) \times \vec{a}''(t) \|}{\| \vec{a}'(t) \|} \quad (23)$$

Exercise 6 Derive equation (23) from equation (22).

\(^{11}\text{The rectangle } \mathcal{R} \text{ also makes this clear.}\)
(d): Solving (23) for \( \kappa(t) \) gives what is in most cases the best way to compute curvature:

\[
\kappa(t) = \frac{\|r''(t) \times r'''(t)\|}{\|r''(t)\|^3}.
\] (24)

(e): For planar curves, one can simplify equation (19) to equation (25).

**Exercise 7** For a planar curve \( \bar{r}(t) = (f(t), g(t), 0) \), show that (24) reduces to

\[
\kappa(t) = \frac{\left| \det \begin{pmatrix} f'(t) & g'(t) \\ f''(t) & g''(t) \end{pmatrix} \right|}{\left( \sqrt{(f'(t))^2 + (g'(t))^2} \right)^3}.
\] (25)

(f): When the planar curve is the graph of a function \( y = g(t) \), equation (25) can be further simplified to equation (26).

**Exercise 8** Use equation (25) to show that for the graph of a twice-differentiable function \( y = g(t) \), the curvature at the point \((t, g(t))\) is given by

\[
\kappa(t) = \frac{|g''(t)|}{\left(1 + (g'(t))^2\right)^{3/2}}.
\] (26)
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</tr>
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<td>9.</td>
<td>$\text{speed}'(t) = a_{T}(t) = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{|\vec{r}'(t)|}$</td>
<td>(21)</td>
<td>p.8</td>
</tr>
<tr>
<td>10.</td>
<td>$a_{R}(t) = |\vec{r}'(t)|^2 \kappa(t) = |\vec{r}'(t) \times \vec{r}''(t)| |\vec{r}'(t)|$</td>
<td>(23)</td>
<td>p.8</td>
</tr>
<tr>
<td>11.</td>
<td>$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$</td>
<td>(24)</td>
<td>p.9</td>
</tr>
<tr>
<td>12.</td>
<td>$\kappa(t) = \left</td>
<td>\det \begin{pmatrix} f'(t) &amp; g'(t) \ f''(t) &amp; g''(t) \end{pmatrix} \right</td>
<td>\left( \sqrt{(f'(t))^2 + (g'(t))^2} \right)^3$</td>
</tr>
<tr>
<td>13.</td>
<td>$\kappa(t) = \frac{</td>
<td>g''(t)</td>
<td>}{(\sqrt{1 + (g'(t))^2})^3}$</td>
</tr>
</tbody>
</table>