The Law of Cosines and the Dot Product

In this handout, I will remind you what the Law of Cosines says and why it is true; and then I will use the Law of Cosines to establish the main property of the dot product.

**Theorem 1.** Let \( \triangle ABC \) be any triangle, with side \( a \) opposite \( \angle A \), side \( b \) opposite \( \angle B \), and side \( c \) opposite \( \angle C \). Then

\[
c^2 = a^2 + b^2 - 2ab \cos C.
\]

**Proof.** Place the triangle in the \((x, y)\)-plane with vertex \( C \) at the origin and side \( b \) along the positive \( x \)-axis. Then vertex \( A \) has coordinates \((b, 0)\), and vertex \( B \) has polar coordinates \((r, \theta) = (a, C)\) and hence rectangular coordinates \((a \cos C, a \sin C)\). Therefore, \( c^2 \) can be obtained by using the distance formula and simplifying:

\[
c^2 = (a \cos C - b)^2 + a^2 \sin^2 C = (a^2 \cos^2 C - 2ab \cos C + b^2) + a^2 \sin^2 C =
\]

\[
\begin{align*}
&\text{\downarrow \hspace{2cm} rearrange terms} \hspace{2cm} \downarrow \\
&= a^2 \cos^2 C + a^2 \sin^2 C + b^2 - 2ab \cos C = a^2 \left( \cos^2 C + \sin^2 C \right) + b^2 - 2ab \cos C = a^2 + b^2 - 2ab \cos C. \\
\end{align*}
\]

**Theorem 2.** Let \( \vec{u} \) and \( \vec{v} \) be any nonzero vectors in either \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \). Let \( \theta \) be the nonreflex angle between them. Then

\[
\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta.
\]

**Proof.** If \( \vec{u} \) is a scalar multiple of \( \vec{v} \)—say, \( \vec{u} = c\vec{v} \)—then \((*)\) can easily be checked by algebra. (There are two cases: \( c > 0 \leftrightarrow \theta = 0 \) and \( c < 0 \leftrightarrow \theta = \pi \).) Otherwise, \( \vec{u} \) and \( \vec{v} \) (in standard position) run along two sides of a triangle, and the vector \((\vec{u} - \vec{v})\) runs along the third side (see diagram). Now, there are two expressions for the square of the length of the third side; this theorem is proved by equating them.

The first expression is just the Law of Cosines applied to this triangle (note that \( \angle C \) is just \( \theta \)):

\[
\|\vec{u} - \vec{v}\|^2 = c^2 = a^2 + b^2 - 2ab \cos \theta = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2 \|\vec{u}\| \|\vec{v}\| \cos \theta.
\]

The second expression is obtained by using properties of dot product developed in class to simplify \(\|\vec{u} - \vec{v}\|^2\):

\[
\|\vec{u} - \vec{v}\|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) =
\]

\[
\begin{align*}
&\text{\downarrow \hspace{2cm} "multiply out" \hspace{2cm} \downarrow} \\
&\vec{u} \cdot \vec{u} - 2 \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} = \|\vec{u}\|^2 - 2 \vec{u} \cdot \vec{v} + \|\vec{v}\|^2. \\
\end{align*}
\]

Finally, we equate the expressions on the right sides of equations (1) and (2):

\[
\|\vec{u}\|^2 + \|\vec{v}\|^2 - 2 \|\vec{u}\| \|\vec{v}\| \cos \theta = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2 \vec{u} \cdot \vec{v} \\
\]

\[
\begin{align*}
&\text{\downarrow \hspace{2cm} simplify and cancel \hspace{2cm} \downarrow} \\
&\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta. \\
\end{align*}
\]