Moment of Inertia

1 Introduction.

You are familiar with the fact that two simultaneously dropped objects will fall side by side, regardless of their weights.\(^1\) You may be surprised to learn, however, that if you release two round objects simultaneously at the top of an inclined plane, they will not necessarily roll down the plane side by side. Roughly speaking, the reason for the difference is this. In the first case, the greater force of gravity on a heavier object (governed by gravitational mass) is exactly cancelled by the decreased effect of gravity (governed by inertial mass) on the acceleration of the object; the cancellation is exact because gravitational mass equals inertial mass. In the second case, though, gravity accelerates the objects by making them roll faster and faster, and cancellation is not exact, because more than just inertial mass alone governs an object’s resistance to changes in its spin speed. This handout explains how resistance to spin-speed change—called moment of inertia—is defined and calculated.

2 Two Parallel Systems.

We need to compare and eventually connect the theory of linear motion with the largely parallel theory of spinning motion (called angular or rotational motion). The beginnings of the two theories are almost identical.

<table>
<thead>
<tr>
<th>Linear motion:</th>
<th>Rotational motion:</th>
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<tbody>
<tr>
<td>• <strong>Setting:</strong> A very small object of mass (m) free to move in space.</td>
<td>• <strong>Setting:</strong> A very small object of mass (m) constrained to the circle (x^2 + y^2 = a^2) in the plane (but free to move around on the circle).</td>
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<tr>
<td>• (r(t)) is the position of the object at time (t).</td>
<td>• (\theta(t)) is the angular position of the object at time (t), so that the object’s position vector is (r(t) = a(\cos(\theta(t)), \sin(\theta(t)))).</td>
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<tr>
<td>• (v(t) = r'(t)) is the velocity of the object at time (t).</td>
<td>• (\omega(t) = \theta'(t)) is the angular velocity of the object at time (t). (</td>
</tr>
<tr>
<td>• (a(t) = v'(t)) is the acceleration of the object at time (t).</td>
<td>• (\alpha(t) = \omega'(t)) is the angular acceleration of the object at time (t).</td>
</tr>
</tbody>
</table>

The strategy for identifying moment of inertia is to develop an angular-motion analogue of the equation

\[
\mathbf{F}(t) = m\mathbf{a}(t) \implies \|\mathbf{F}(t)\| = m\|\mathbf{a}(t)\|.
\]

\(^1\)in the absence of air resistance. Assume vacuum conditions throughout this handout.
Observe that in equation (1):

- the vector \( \mathbf{F}(t) \) effects a change in the object’s velocity \( \mathbf{v}(t) \);
- the magnitude in the change to \( \mathbf{v}(t) \) is proportional to the magnitude of the force; and
- the constant of proportionality \( m \) measures the extent to which the object resists this change: the larger \( m \) is, the smaller the acceleration induced by a given force will be.

The hope is to identify an angular-acceleration equation of the form

\[
\tau(t) = I\alpha(t),
\]

where

- \( \tau(t) \) is something that effects a change in the object’s angular velocity \( \omega(t) \);
- the magnitude in the change to \( \omega(t) \) is proportional to the magnitude of \( \tau(t) \); and
- the constant of proportionality \( I \) (which will be the moment of inertia) measures the extent to which the spinning object resists this change.

We will first identify what \( \tau(t) \) is and then use \( \tau(t) \) to isolate the moment of inertia \( I \).

3 The Torque: \( \tau(t) \).

You encountered this quantity when center of gravity was being discussed. You exert a torque on a seesaw when you sit on it: the torque you exert is either clockwise or counterclockwise, and the magnitude of the torque is the product of your weight and your distance from the fulcrum of the seesaw. If you think of the seesaw as an object that is free to spin around its fulcrum, then a torque will change the object’s angular velocity.

Now suppose a force \( \mathbf{F}(t) \) is exerted on the object on the circle \( x^2 + y^2 = a^2 \). \( \mathbf{F}(t) \) can be resolved into the sum of a tangential force \( F_{\text{tan}}(t) \) and a radial force \( F_{\text{rad}}(t) \)

\[
\mathbf{F}(t) = F_{\text{rad}}(t) + F_{\text{tan}}(t)
\]

(see diagram). \( F_{\text{rad}}(t) \) will be cancelled by whatever force is constraining the object to the circle, whereas \( F_{\text{tan}}(t) \) generates a torque \( \tau(t) \), of magnitude \( a\|F_{\text{tan}}(t)\| \), which the object’s angular velocity to change.

3.1 The Sign of \( \tau(t) \).

Let us focus on the nature of the change to \( \omega(t) \) brought about by \( \tau(t) \). Say that (as in the diagram), the torque is pointed in the counterclockwise direction.

- If the object is at rest, it will start to rotate counterclockwise, so that \( \omega(t) \) goes from zero to positive.
- If the object is already spinning counterclockwise—that is, if \( \theta'(t) = \omega(t) \) is positive—then \( \tau(t) \) will cause the object to spin faster, so that \( \omega(t) \) increases.
- If the object is already spinning clockwise—that is, if \( \theta'(t) = -\omega(t) \) is negative—then \( \tau(t) \) will cause the object to slow down, so that \( |\omega(t)| \) decreases but \( \omega(t) \) increases (becomes less negative).
- In all three cases, since \( \omega(t) \) is increasing, \( \alpha(t) = \omega'(t) \) will be positive.

Similarly: if the torque is pointed in a clockwise direction, then \( \alpha(t) \) will be negative. Therefore, in order to ensure that \( I > 0 \) in equation (2), one takes counterclockwise torques to be positive and clockwise torques to be negative.
Putting all this together gives the equation
\[
\tau(t) = \begin{cases} 
+a \|F_{\text{tan}}(t)\|, & \text{if the torque is counterclockwise;} \\
-a \|F_{\text{tan}}(t)\|, & \text{if the torque is clockwise.}
\end{cases}
\] (3)

4 The Moment of Inertia.

In order to pin down what \( I \) is, we need to find a point of contact between the two systems. One way to do this is to calculate the kinetic energy of the spinning point mass in two different ways and to compare the results. Recall that when force \( \mathbf{F} \) moves an object through a displacement \( \mathbf{d}s \), it does work
\[
dW = \mathbf{F} \cdot \mathbf{d}s
\]
on the object, which in the absence of friction, air resistance, etc. is realized as a change to the kinetic energy of the object.

In order to simplify the discussion, let us suppose
- that the point mass is at rest at time \( t = 0 \) and
- that for interval \( 0 \leq t \leq b \), it is acted on by a force
\[
\mathbf{F}(t) = F_{\text{rad}}(t) + F_{\text{tan}}(t),
\] (4)

where \( F_{\text{tan}}(t) \) points in a counterclockwise direction.

4.1 The Linear Calculation.

In an infinitesimal time interval \([t, t + dt]\), we have
\[
dW = \mathbf{F}(t) \cdot \mathbf{d}s = \mathbf{F}(t) \cdot dt \mathbf{v}(t)
\]
(Using equation (1) \( \rightarrow \) \( = m \mathbf{v}'(t) \cdot \mathbf{v}(t) dt \))

so that

\[
\begin{align*}
\text{kinetic energy at time } b & = \int_0^b dW \\
& = \int_0^b m \mathbf{v}'(t) \cdot \mathbf{v}(t) dt \\
& = m \int_0^b \mathbf{v}'(t) \cdot \mathbf{v}(t) dt \\
& = m \int_0^b \left( \frac{d}{dt} \left[ \frac{1}{2} \mathbf{v}(t) \cdot \mathbf{v}(t) \right] \right) dt \\
& = m \int_0^b \left( \frac{d}{dt} \left[ \frac{1}{2} \mathbf{v}(t) \cdot \mathbf{v}(t) \right] \right) dt \\
& = m \frac{1}{2} (\mathbf{v}(b) \cdot \mathbf{v}(b)) \\
\text{kinetic energy at time } b & = \frac{1}{2} m \| \mathbf{v}(b) \|^2.
\end{align*}
\] (5) (6)
4.2 The Angular Calculation.

In an infinitesimal time interval \([t, t + dt]\), we have

\[
dW = \mathbf{F}(t) \cdot \overrightarrow{ds}
\]

(Equation (4) \(\rightarrow\))

\[
= \mathbf{F}_{\text{rad}}(t) \cdot \overrightarrow{ds} + \mathbf{F}_{\text{tan}}(t) \cdot \overrightarrow{ds}
\]

\[
= 0 + (\|\mathbf{F}_{\text{tan}}(t)\|)(\|\overrightarrow{ds}\|)
\]

(equation (3), top line \(\rightarrow\))

\[
(\|\overrightarrow{ds}\| = a|d\theta|\text{ and } d\theta \geq 0 \rightarrow) = \frac{\tau(t)}{a} (a \, d\theta)
\]

(cancel the \(a\)'s; \(d\theta = \omega(t) dt \rightarrow\))

\[
= \tau(t) \omega(t) dt
\]

(equation (2) \(\rightarrow\))

\[
= I\alpha(t) \omega(t) dt.
\]

Therefore,

\[
\text{kinetic energy at time } b = \int_0^b dW
\]

\[
= \int_0^b I\alpha(t) \omega(t) dt
\]

\[
= I \int_0^b \left( \frac{d}{dt} \left[ \frac{1}{2} \omega^2(t) \right] \right) dt
\]

\[
\text{kinetic energy at time } b = \frac{1}{2} I (\omega(b))^2.
\]

(7)

4.3 Putting the Pieces Together.

Setting the two kinetic energy expressions (refke1) and (7) equal to each other gives us

\[
\frac{1}{2} m \|\mathbf{v}(b)\|^2 = \frac{1}{2} I (\omega(b))^2.
\]

(8)

In order to solve (8) for \(I\), one more observation is needed: since \(r(t) = a(\cos(\theta(t)), \sin(\theta(t)))\), we have that

\[
\mathbf{v}(t) = r'(t) = a\theta'(t) (-\sin(\theta(t)), \cos(\theta(t))),
\]

so that

\[
\|\mathbf{v}(t)\| = a|\theta'(t)| = a|\omega(t)|.
\]

(9)

Substituting (9) into (8) then gives

\[
\frac{1}{2} m (a\omega(b))^2 = \frac{1}{2} I (\omega(b))^2,
\]

(10)

and solving (10) for \(I\) gives the formula we are seeking:

\[
I = ma^2.
\]