Some Path Limit Examples

\[ f(x, y) = \frac{xy^2}{x^2 + y^4} \quad g(x, y) = \frac{xy^2}{x^2 + y^6} \]

Along vertical paths:

\[ \lim_{y \to 0} f(0, y) = \lim_{y \to 0} \frac{0}{y^4} = \lim_{y \to 0} 0 = 0 \]
\[ \lim_{y \to 0} g(0, y) = \lim_{y \to 0} \frac{0}{y^6} = \lim_{y \to 0} 0 = 0 \]

Along non-vertical linear paths \( y = mx \)

\[ \lim_{x \to 0} f(x, mx) = \lim_{x \to 0} \frac{x(mx)^2}{x^2 + (mx)^4} = \lim_{x \to 0} \frac{m^2 x^3}{x^2(1 + m^4 x^2)} \]
\[ = \lim_{x \to 0} \frac{m^2 x}{1 + m^4 x^2} = 0 = 0 \]

\[ \lim_{x \to 0} g(x, mx) = \lim_{x \to 0} \frac{x(mx)^2}{x^2 + (mx)^6} = \lim_{x \to 0} \frac{m^2 x^3}{x^2(1 + m^6 x^4)} \]
\[ = \lim_{x \to 0} \frac{m^2 x}{1 + m^6 x^4} = 0 = 0 \]

However...

Along the path \( y = \sqrt{x} \)

\[ \lim_{x \to 0^+} f(x, \sqrt{x}) = \lim_{x \to 0^+} \frac{x(\sqrt{x})^2}{x^2 + (\sqrt{x})^4} \]
\[ = \lim_{x \to 0^+} \frac{x^{3/2}}{x^2 + x^{2}} = \lim_{x \to 0^+} \frac{x^{3/2}}{2x^2} \]
\[ = \lim_{x \to 0^+} \frac{1}{2} = \frac{1}{2} \]

So the overall limit DNE

Along the path \( y = \frac{\sqrt{x}}{x} \)

\[ \lim_{x \to 0^+} \frac{x^{5/3}}{x^2 + (\sqrt{x})^6} = \lim_{x \to 0^+} \frac{x^{5/3}}{x^2 + x^2} \]
\[ = \lim_{x \to 0^+} \frac{x^{5/3}}{2x^2} = \lim_{x \to 0^+} \frac{1}{2^{5/3} \sqrt{x}} = \infty \]

In this case, not only does the overall limit DNE, but the function \( g(x, y) \) takes arbitrarily large values in every disk around the origin.