THE DERIVATIVE OF A FUNCTION OF TWO VARIABLES

As discussed in class, the generalization of the notions of differentiability and derivatives from Calc I functions to functions of two variables is much less straightforward than is the generalization from Calc I functions to curves. What we need to do in this case, it turns out, is to focus on the tangent-line aspect of the Calc I derivative. Section I of this handout presents a summary of the relationship of the derivative to the tangent line in the Calc I case; section II develops the analogous notion of "tangent plane" for functions of two variables; section III finds the equation of the only possible tangent plane ("If there is one at all, this is its equation"); and section IV gives conditions under which we are guaranteed that there indeed will exist a tangent plane.

I. The tangent line. Let \( y = f(x) \) be a scalar-valued function of one variable defined near \( x_0 \), and let \( \Delta x := x - x_0 \). As you know, \( f \) is differentiable at \( x_0 \) if and only if \( \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \) exists; and if this limit does exist, then its value is the derivative \( f'(x_0) \):

\[
f'(x_0) := \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.
\]

(1)

As you also know, the line that is tangent to the graph of \( f(x) \) at \( (x_0, f(x_0)) \) is then defined to be the line containing point \( (x_0, f(x_0)) \) whose slope is \( f'(x_0) \). We need to examine this definition more closely. The equation of any nonvertical line that contains the point \( (x_0, f(x_0)) \) can be written in the form

\[ y = f(x_0) + m(x - x_0), \]

where \( m \) is the slope of the line. Of all of these, the tangent line is the one for which \( m = f'(x_0) \):

\[ y = f(x_0) + f'(x_0)(x - x_0) \]

or equivalently

\[ L_{x_0}(x) = f(x_0) + f'(x_0)(x - x_0). \]

(2)

Example. Let \( g(x) = \sqrt{x} \) and let \( x_0 = \frac{1}{4} \). Then

\[ g\left(\frac{1}{4}\right) = \frac{1}{2}; \]

\[ g'(x) = \frac{1}{2\sqrt{x}}; \]

\[ g'\left(\frac{1}{4}\right) = \frac{1}{2\sqrt{\frac{1}{4}}} = 1; \]

\[ L_{\frac{1}{4}}(x) = \frac{1}{2} + 1 \cdot \left(x - \frac{1}{4}\right). \]

The property that makes the tangent line special this: it turns out that as you zoom in closer and closer to the point \( (x_0, f(x_0)) \), the graph of \( y = f(x) \) becomes less and less distinguishable from the graph of \( y = L_{x_0}(x) \). On the last page of this handout, you can see what the graphs of \( g(x) = \sqrt{x} \) and \( L_{\frac{1}{4}}(x) = x + \frac{1}{4} \) look like as the magnification is increased.

The way to express this property mathematically is through the formula

\[
\lim_{\Delta x \to 0} \left| \frac{f(x_0 + \Delta x) - L_{x_0}(x_0 + \Delta x)}{\Delta x} \right| = 0,
\]

(3)

because the zooming process corresponds to multiplying by \( \frac{1}{\Delta x} \) (which is very large when \( x \) is close to \( x_0 \)).

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Furthermore, Equation (3) can be derived from equations (1) and (2) as follows. If you subtract \( f'(x_0) \) from both sides of Equation (1), you get

\[
\left( \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \right) - f'(x_0) = 0,
\]

and by the limit law \( \lim_{t \to a} C = C \), this can be rewritten

\[
\left( \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \right) - \lim_{\Delta x \to 0} f'(x_0) = 0.
\]

Then, by the limit law \( \lim_{t \to a} H(t) - K(t) = \lim_{t \to a} (H(t) - K(t)) \), this can be transformed into

\[
\lim_{\Delta x \to 0} \left( \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} - f'(x_0) \right) = 0.
\]

The rest is arithmetic:

\[
\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0) - f'(x_0)\Delta x}{\Delta x} = 0
\]

\[
\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0) + f'(x_0)\Delta x}{\Delta x} = 0
\]

\[
\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - L_{x_0}(x_0 + \Delta x)}{\Delta x} = 0
\]

\[
\lim_{\Delta x \to 0} \frac{|f(x_0 + \Delta x) - L_{x_0}(x_0 + \Delta x)|}{|\Delta x|} = |0| = 0.
\]

**Exercise 1.** Prove that this process is reversible; that is, prove the following. Suppose, for some function \( f \) and some point \( x_0 \), that there is a line

\[
L(x) = f(x_0) + m\Delta x
\]

for which

\[
\lim_{\Delta x \to 0} \frac{|f(x_0 + \Delta x) - L(x_0 + \Delta x)|}{|\Delta x|} = 0.
\]

Then \( f \) is in fact differentiable at \( x_0 \), and \( f'(x_0) = m \). (Hint: the above sequence of equations can be followed in the reverse direction.)

**II. The tangent plane and the definition of differentiability.** It is the property expressed in equation (3) that can be adapted to the present context. Let \( z = f(x, y) \) be a function of two variables defined near \((x_0, y_0)\), let \( z_0 = f(x_0, y_0) \), and let

\[
\begin{cases} 
\Delta x := x - x_0 \\
\Delta y := y - y_0.
\end{cases}
\]

Note that the equation of any nonvertical plane that contains the point \((x_0, y_0, z_0)\) can be written in the form

\[
z = f(x_0, y_0) + a(x-x_0) + b(y-y_0)
\]

or equivalently

\[
L_{(x_0, y_0)}(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0) + a\Delta x + b\Delta y
\]
for some constants $a$ and $b$. We will define the tangent plane to the graph of $f$ at $(x_0, y_0, f(x_0, y_0))$ of the plane\(^1\) which possesses a zooming-in property like the one for tangent lines captured by Equation (3):

$$\lim_{(\Delta x, \Delta y) \to (0, 0)} \frac{|f(x_0 + \Delta x, y_0 + \Delta y) - L_{(x_0, y_0)}(x_0 + \Delta x, y_0 + \Delta y)|}{\| (\Delta x, \Delta y) \|} = 0. \tag{5}$$

Of course, it is possible for there to be no such plane; we will say that $f$ is differentiable at $(x_0, y_0)$ if the graph of $f$ has a tangent plane at the point $(x_0, y_0, f(x_0, y_0))$.\(^2\)

Equation (5) does indeed capture the two-dimensional zooming property; it says that as you zoom in closer and closer to the point $(x_0, y_0, f(x_0, y_0))$—by multiplying by $\frac{1}{\| (\Delta x, \Delta y) \|}$—the surface $z = f(x, y)$ becomes less and less distinguishable from the plane $z = L_{(x_0, y_0)}(x, y)$. (This phenomenon is what made people believe for so long that the earth is flat.)

III. The only candidate. So, if $f$ has a tangent plane at $(x_0, y_0, z_0)$, what might its equation be? Here is a geometric way to answer this question.\(^3\) The strategy will be to use the methods of Chapter 13 to find two nonparallel vectors $\vec{v}_1$ and $\vec{v}_2$ in the plane, since then the cross product $(\vec{v}_2 \times \vec{v}_1)$ will be a normal vector for the plane. To simplify the explanation, I will assume that the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ exist in some disc around $(x_0, y_0)$, but this assumption is not a necessary condition.

To find $\vec{v}_1$, consider the curve

$$\vec{r}_1(t) = (t, y_0, f(t, y_0)),$$

which follows a path that lies in the graph of $z = f(x, y)$ and also in the plane $y = y_0$. The derivative of this curve is

$$\vec{r}'_1(t) = (1, 0, f_x(x_0, y_0)),$$

so the vector

$$\vec{v}_1 := \vec{r}'_1(x_0) = (1, 0, f_x(x_0, y_0)),$$

will be tangent to the curve $\vec{r}_1$ at $(x_0, y_0, z_0)$ and hence will lie in the tangent plane. A precisely analogous argument applied to

$$\vec{r}_2(t) = (x_0, t, f(x_0, t)),$$

shows that the vector

$$\vec{v}_2 := \vec{r}'_2(y_0) = (0, 1, f_y(x_0, y_0))$$

will also lie in the tangent plane. Since these vectors are not parallel (How do I know this??), the vector $(\vec{v}_2 \times \vec{v}_1)$ is a normal vector for the tangent plane. By a simple calculation,

$$\vec{v}_2 \times \vec{v}_1 = \begin{pmatrix} 0, 1, f_y(x_0, y_0) \end{pmatrix} \times \begin{pmatrix} 1, 0, f_x(x_0, y_0) \end{pmatrix} = \begin{pmatrix} f_y(x_0, y_0), f_y(x_0, y_0), -1 \end{pmatrix},$$

so the equation of the tangent plane, if there is one, would have to be

$$(f_x(x_0, y_0), f_y(x_0, y_0), -1) \cdot (x - x_0, y - y_0, z - f(x_0, y_0)) = 0,$$

which can be rewritten in the form of Equation (4):

$$\begin{cases}
  z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
  L_{(x_0, y_0)}(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y
\end{cases} \tag{6}$$

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\(^1\) I am using the word "the" here because there is only one such plane, if there even is one at all (see Section III).

\(^2\) I should note that this is not exactly how the definition in the text reads (p.918); but the definition here is easier to motivate (and is equivalent to the text's definition).

\(^3\) Exercise 2 gives an algebraic approach.
Thus, if there is a tangent plane at all, it must be the plane given by (6).

**Exercise 2. (Hard!)** (This exercise backs the cross-product derivation of equation (6) with limit computations, which are on firmer logical ground.) Let

\[
L(x_0, y_0)(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0) + a\Delta x + b\Delta y
\]

for some numbers \(a\) and \(b\). Show that if equation (5) holds for this function \(L(x_0, y_0)\), then

\[
\begin{align*}
\left\{ f_x(x_0, y_0) \right\} & \text{ exists and equals } a, \\
\left\{ f_y(x_0, y_0) \right\} & \text{ exists and equals } b.
\end{align*}
\]

*Hint:* Begin with the definition of \(f_x\) (respectively \(f_y\)) on p. 902 of the text.

**IV. When will there actually be a tangent plane?** The upshot of the discussion of in Section I was that for a Calc I function \(y = f(x)\), conditions (\(\alpha\)) and (\(\beta\)) below are completely equivalent:

\[
\text{the limit } f'(x_0) := \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \text{ will exist}
\]

if, and only if,

\[
\text{the line } L_{x_0}(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x \text{ satisfies Equation (3)}.
\]

For a Calc III function \(z = f(x, y)\), unfortunately, the corresponding two conditions are not equivalent: the Calc III version of (\(\beta\)) (Equation (5)) will imply the Calc III version of (\(\alpha\))—this is what Exercise (2) is saying—but the Calc III version of (\(\alpha\)) does not imply Equation (5). (Exercises #45 and #46 on p.924 of your text present a case in which the partial derivatives exist but the tangent plane does not.)

Additional assumptions, however, fix the problem. If we know not only that \(f_x(x_0, y_0)\) and \(f_y(x_0, y_0)\) exist but also that \(f_x\) and \(f_y\) exist everywhere in a disk around \((x_0, y_0)\) and are continuous at \((x_0, y_0)\), then in fact \(L(x_0, y_0)\) will satisfy Equation (5). I will give an “almost-proof” of this assertion and afterwards indicate a slight logical gap in the argument.

**Theorem.** If \(f_x(x, y)\) and \(f_y(x, y)\) exist in a disk around \((x_0, y_0)\) and are continuous at \((x_0, y_0)\), then

\[
\lim_{(\Delta x, \Delta y) \to (0, 0)} \frac{|f(x_0 + \Delta x, y_0 + \Delta y) - L(x_0, y_0)(x_0 + \Delta x, y_0 + \Delta y)|}{\| (\Delta x, \Delta y) \|} = 0.
\]

**Almost-proof.** *Step 1.* I will transform Equation (5) into a more convenient form. I will begin by substituting (6) into (5) to get

\[
\lim_{(\Delta x, \Delta y) \to (0, 0)} \frac{|f(x_0 + \Delta x, y_0 + \Delta y) - [f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y]|}{\| (\Delta x, \Delta y) \|} = 0;
\]

then, I will rearrange terms to get

\[
\lim_{(\Delta x, \Delta y) \to (0, 0)} \frac{|(f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)) - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y|}{\| (\Delta x, \Delta y) \|} = 0.
\]

*Finally, I will add and subtract \(f(x_0 + \Delta x, y_0)\) to \(*\) in Equation (7) to get

\[
\lim_{(\Delta x, \Delta y) \to (0, 0)} \frac{|(f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)) + (f(x_0 + \Delta x, y_0) - f(x_0, y_0)) - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y|}{\| (\Delta x, \Delta y) \|} = 0
\]

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Step 2. I will apply the Mean Value Theorem to expressions (A) and (B) in Equation (8) (see diagram).

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{|f_y(x_0 + \Delta x, y^*)\Delta y + f_z(x^*, y_0)\Delta z - f_z(x_0, y_0)\Delta z - f_y(x_0, y_0)\Delta y|}{\| \Delta x, \Delta y \|} = 0$$

or, with terms rearranged,

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{|(f_z(x^*, y_0) - f_z(x_0, y_0))\Delta x + (f_y(x_0 + \Delta x, y^*) - f_y(x_0, y_0))\Delta y|}{\| \Delta x, \Delta y \|} = 0. \tag{9}$$

Step 3. I will make preparations to apply the Squeeze Theorem by overestimating Expression (C) in Equation (9). First, by the rule $|a + b| \leq |a| + |b|$, we get

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{|f_z(x^*, y_0) - f_z(x_0, y_0)|}{\| \Delta x, \Delta y \|} + \frac{|f_y(x_0 + \Delta x, y^*) - f_y(x_0, y_0)|}{\| \Delta x, \Delta y \|} \leq \sqrt{(\Delta x)^2 + (\Delta y)^2} = \| \Delta x, \Delta y \| \tag{10}$$

Next, note that

$$0 \leq \left\{ \frac{(\Delta x)^2}{(\Delta y)^2} \right\} \leq (\Delta x)^2 + (\Delta y)^2 \implies 0 \leq \left\{ \frac{|\Delta x|}{|\Delta y|} \right\} \leq \sqrt{(\Delta x)^2 + (\Delta y)^2} = \| \Delta x, \Delta y \|;$$

this allows us to replace larger with smaller denominators in (10) and to simplify:

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{|f_z(x^*, y_0) - f_z(x_0, y_0)|}{\| \Delta x, \Delta y \|} + \frac{|f_y(x_0 + \Delta x, y^*) - f_y(x_0, y_0)|}{\| \Delta x, \Delta y \|} \leq \frac{|f_z(x^*, y_0) - f_z(x_0, y_0)|}{|\Delta x|} + \frac{|f_y(x_0 + \Delta x, y^*) - f_y(x_0, y_0)|}{|\Delta y|} \tag{11}$$

Step 4. Apply the Squeeze Theorem. The proof so far has shown that

$$0 \leq \frac{|f(x + \Delta x, y + \Delta y) - L(x_0, y_0)(x + \Delta x, y + \Delta y)|}{\| \Delta x, \Delta y \|} \leq \frac{|f_x(x^*, y_0) - f_x(x_0, y_0)|}{|\Delta x|} + \frac{|f_y(x_0 + \Delta x, y^*) - f_y(x_0, y_0)|}{|\Delta y|}; \tag{12}$$

and since $f_x$ and $f_y$ are both continuous at $(x_0, y_0)$,

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{|f(x^*, y_0) - f(x_0, y_0)| + |f_y(x_0 + \Delta x, y^*) - f_y(x_0, y_0)|}{\| \Delta x, \Delta y \|} = 0.$$

Equation (13) now allows us to apply the Squeeze theorem to Equation (12) to get

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{|f(x, y) - L(x_0, y_0)(x, y)|}{\| \Delta x, \Delta y \|} = 0. \tag{13}$$

The gap in the proof. The problem is this. The definition of $\lim_{(x,y) \to (x_0,y_0)}$ insures that

$$(x, y) \neq (x_0, y_0)$$

so that $\Delta x$ and $\Delta y$ cannot both be zero; but one of them can be zero, provided that the other is not. In this case, the computation in line (11) is illegal. This gap can readily be closed by including separate arguments for these cases.
> f := x \rightarrow \exp(x);
> g := f(x);
> plot([x, x + \frac{1}{x}, x = 0.3, color=black]);